Filling multiples of embedded curves and quantifying nonorientability

Robert Young
New York University

December 2017

Filling multiples of embedded curves

If T is an integral 1-cycle (i.e., union of oriented closed curves) in $\mathbb{R}^n$, let $\mathrm{FA}(T)$ (*filling area*) be the minimal area of an integral 2-chain with boundary T.

Filling multiples of embedded curves

If T is an integral 1-cycle (i.e., union of oriented closed curves) in $\mathbb{R}^n$, let $\mathrm{FA}(T)$ (*filling area*) be the minimal area of an integral 2-chain with boundary T.

How is $\mathrm{FA}(T)$ related to $\mathrm{FA}(2T)$?

How is FA(T) related to FA($2T$)?

For all T, $\mathrm{FA}(2T) \leq 2\,\mathrm{FA}(T)$.

How is $\mathrm{FA}(T)$ related to $\mathrm{FA}(2T)$?

For all T, $\mathrm{FA}(2T) \leq 2\,\mathrm{FA}(T)$.

- $n = 2$: If T is a curve in $\mathbb{R}^2$, then $\mathrm{FA}(2T) = 2\,\mathrm{FA}(T)$.

How is FA(T) related to FA($2T$)?

For all T, $\mathrm{FA}(2T) \leq 2\,\mathrm{FA}(T)$.

- $n = 2$: If T is a curve in $\mathbb{R}^2$, then $\mathrm{FA}(2T) = 2\,\mathrm{FA}(T)$.
- $n = 3$: If T is a curve in $\mathbb{R}^3$, then $\mathrm{FA}(2T) = 2\,\mathrm{FA}(T)$. (Federer, 1974)

How is $\mathrm{FA}(T)$ related to $\mathrm{FA}(2T)$?

For all T, $\mathrm{FA}(2T) \leq 2\,\mathrm{FA}(T)$.

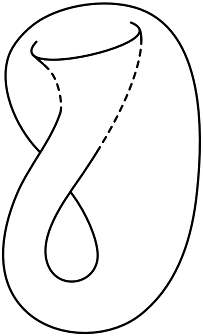
- $n = 2$: If T is a curve in $\mathbb{R}^2$, then $\mathrm{FA}(2T) = 2\,\mathrm{FA}(T)$.
- $n = 3$: If T is a curve in $\mathbb{R}^3$, then $\mathrm{FA}(2T) = 2\,\mathrm{FA}(T)$. (Federer, 1974)
- $n = 4$: There is a curve $T \in \mathbb{R}^4$ such that

$$\mathrm{FA}(2T) \leq 1.52\,\mathrm{FA}(T)$$

  (L. C. Young, 1963)

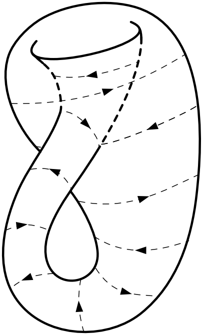
L. C. Young's example

Let K be a Klein bottle

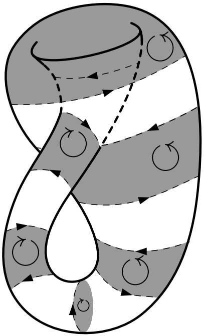


L. C. Young's example

Let K be a Klein bottle and let T be the sum of $2k+1$ loops in alternating directions.

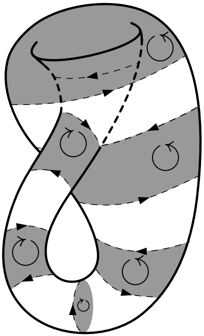


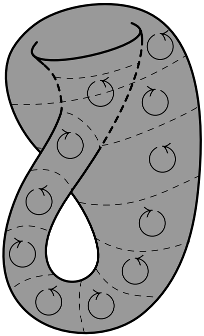
L. C. Young's example



- T can be filled with k bands and one extra disc D
- $\mathrm{FA}(T) \approx \frac{\text{area } K}{2} + \text{area } D$

L. C. Young's example

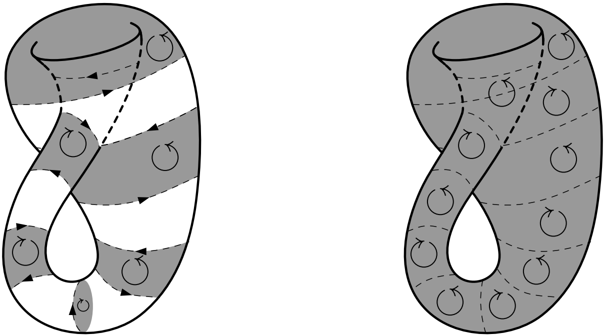




- T can be filled with k bands and one extra disc D
- $\mathrm{FA}(T) \approx \frac{\text{area } K}{2} + \text{area } D$

- $2T$ can be filled with $2k+1$ bands
- $\mathrm{FA}(2T) \approx \text{area } K$

L. C. Young's example



- T can be filled with k bands and one extra disc D
- $\mathrm{FA}(T) \approx \frac{\text{area } K}{2} + \text{area } D$
- $2T$ can be filled with $2k+1$ bands
- $\mathrm{FA}(2T) \approx \text{area } K$— less than $2\,\mathrm{FA}(T)$ by $2\,\text{area } D$!

The main theorem

Q: Is $\mathrm{FA}(2T)$ bounded below by a function of $\mathrm{FA}(T)$?

The main theorem

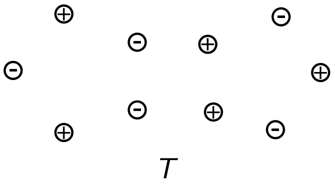
Q: Is $\mathrm{FA}(2T)$ bounded below by a function of $\mathrm{FA}(T)$?

Theorem (Y.)

Yes! For any d, n, there is a $c > 0$ such that if T is a d-cycle in $\mathbb{R}^n$, then $\mathrm{FA}(2T) \geq c\,\mathrm{FA}(T)$.

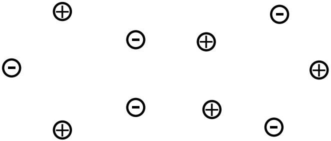
Proving the theorem in dimension 0

Strategy: If B is a filling of $2T$, then "half of B" fills T.

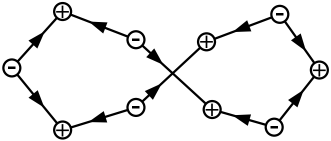


Proving the theorem in dimension 0

Strategy: If B is a filling of $2T$, then "half of B" fills T.



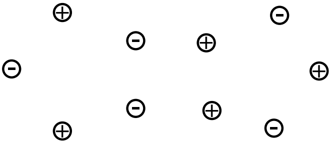
T



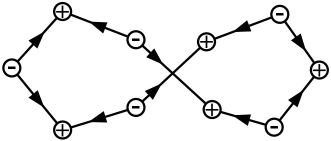
$\partial B = 2T$

Proving the theorem in dimension 0

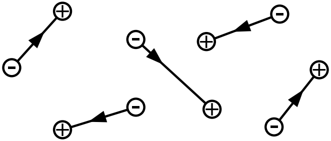
Strategy: If B is a filling of $2T$, then “half of B” fills T.



T



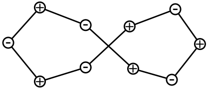
$\partial B = 2T$



“half of B” is a filling of T

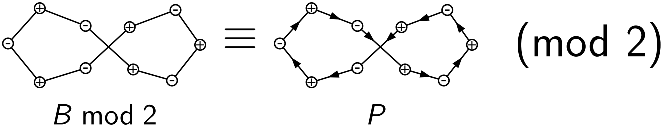
What does “half” mean?

Consider the mod-2 cycle B mod 2.



What does “half” mean?

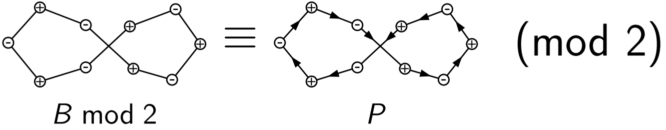
Consider the mod-2 cycle B mod 2.



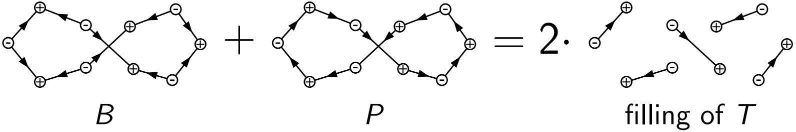
Then B mod 2 is an orientable closed curve with orientation P.

What does "half" mean?

Consider the mod-2 cycle B mod 2.

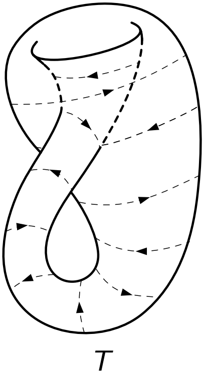


Then B mod 2 is an orientable closed curve with orientation P.



“Half” of the Klein bottle

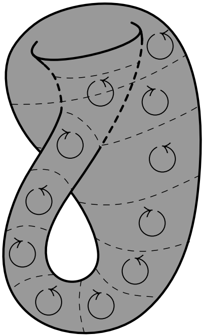
Let T be a cycle



"Half" of the Klein bottle

Let T be a cycle and suppose that

$$\partial B = 2T.$$



B
filling of $2T$

"Half" of the Klein bottle

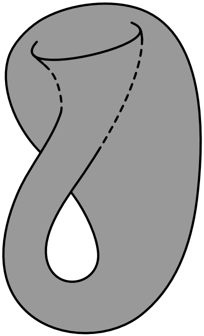
Let T be a cycle and suppose that

$$\partial B = 2T.$$

Then

$$\partial B \equiv 0 \pmod 2,$$

so B mod 2 is a cycle.



B mod 2

"Half" of the Klein bottle

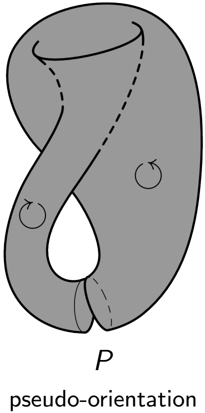
Let T be a cycle and suppose that

$$\partial B = 2T.$$

Then

$$\partial B \equiv 0 \pmod 2,$$

so B mod 2 is a cycle.
If P is an integral cycle such that $B \equiv P \pmod 2$ (a *pseudo-orientation* of B)



“Half” of the Klein bottle

Let T be a cycle and suppose that

$$\partial B = 2T.$$

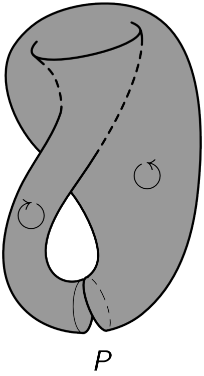
Then

$$\partial B \equiv 0 \pmod 2,$$

so B mod 2 is a cycle.
If P is an integral cycle such that $B \equiv P \pmod 2$ (a *pseudo-orientation* of B), then

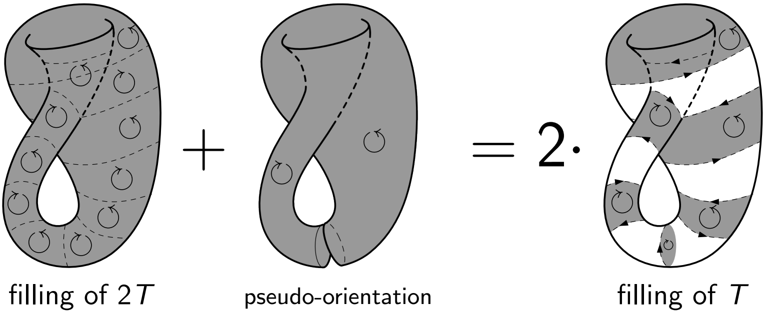
$$B + P \equiv 0 \pmod 2$$

$$\partial \frac{B+P}{2} = \frac{2T+0}{2} = T.$$



pseudo-orientation

The Klein bottle, again



Nonorientability

If A is a mod-2 cycle, define the *nonorientability* of A by

$$\mathrm{NO}(A) = \inf\{\operatorname{mass} P \mid P \text{ is an integral cycle and } P \equiv A \pmod 2\}$$

This measures how hard it is to “lift” A to an integral cycle.

Nonorientability

If A is a mod-2 cycle, define the *nonorientability* of A by

$\mathrm{NO}(A) = \inf\{\mathrm{mass}\, P \mid P \text{ is an integral cycle and } P \equiv A \pmod 2\}$

This measures how hard it is to "lift" A to an integral cycle.
If $\partial B = 2T$, then

$$\mathrm{FV}(T) \le \frac{\mathrm{mass}\, B + \mathrm{NO}(B \bmod 2)}{2}$$

So, to prove that $\mathrm{FV}(T) \lesssim \mathrm{FV}(2T)$, it suffices to show:

Theorem

If A is a mod-2 d-cycle in $\mathbb{R}^n$, then $\mathrm{NO}(A) \lesssim \mathrm{mass}\, A$.

Corollaries

This lets us prove some basic facts about currents and flat chains.

- If $k > 0$ is a positive integer, the multiply-by-k map $f(T) = kT$ on the space of integral flat chains is an embedding with closed image.

Corollaries

This lets us prove some basic facts about currents and flat chains.

- If $k > 0$ is a positive integer, the multiply-by-k map $f(T) = kT$ on the space of integral flat chains is an embedding with closed image.
- If T is a mod-k current, then $T \equiv T_{\mathbb{Z}} \pmod{k}$ for some integral current $T_{\mathbb{Z}}$. Consequently, mod-k currents are a quotient of the integral currents.

Quantifying nonorientability

Theorem

If A is a mod-2 d-cycle in $\mathbb{R}^n$, then $\mathrm{NO}(A) \lesssim \operatorname{mass} A$.

Quantifying nonorientability

Theorem

If A is a mod-2 d-cycle in $\mathbb{R}^n$, then $\mathrm{NO}(A) \lesssim \operatorname{mass} A$*.*

Strategy:

- Find a mod-2 $(d+1)$-chain such that $A = \partial F$.

Quantifying nonorientability

Theorem

If A is a mod-2 d-cycle in $\mathbb{R}^n$, then $\mathrm{NO}(A) \lesssim \operatorname{mass} A$*.*

Strategy:

- Find a mod-2 $(d+1)$-chain such that $A = \partial F$.
- Typically, F is non-orientable. Cut F into orientable pieces to get a lift $F_{\mathbb{Z}}$ of F with integer coefficients.

Quantifying nonorientability

Theorem

If A is a mod-2 d-cycle in $\mathbb{R}^n$, then $\mathrm{NO}(A) \lesssim \operatorname{mass} A$.

Strategy:

- Find a mod-2 $(d+1)$-chain such that $A = \partial F$.
- Typically, F is non-orientable. Cut F into orientable pieces to get a lift $F_{\mathbb{Z}}$ of F with integer coefficients.
- Then $P = \partial F_{\mathbb{Z}}$ is a pseudo-orientation of A.

Quantifying nonorientability

Theorem

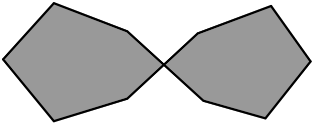
If A is a mod-2 d-cycle in $\mathbb{R}^n$, then $\mathrm{NO}(A) \lesssim \operatorname{mass} A$.

Strategy:

- Find a mod-2 $(d+1)$-chain such that $A = \partial F$.
- Typically, F is non-orientable. Cut F into orientable pieces to get a lift $F_{\mathbb{Z}}$ of F with integer coefficients.
- Then $P = \partial F_{\mathbb{Z}}$ is a pseudo-orientation of A.
- The difference $\operatorname{mass} P - \operatorname{mass} A$ measures how much of F we had to cut.

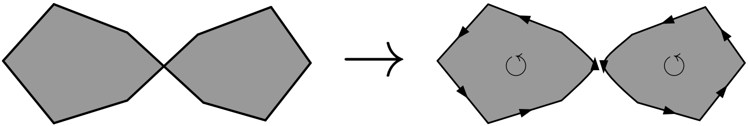
Codimension 1

If A is codimension 1, then A is the boundary of a top-dimensional chain F:



Codimension 1

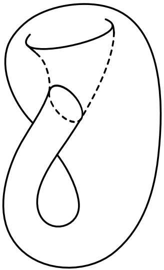
If A is codimension 1, then A is the boundary of a top-dimensional chain F:



F is orientable, so A is orientable and $\mathrm{NO}(A) = \mathrm{mass}(A)$.

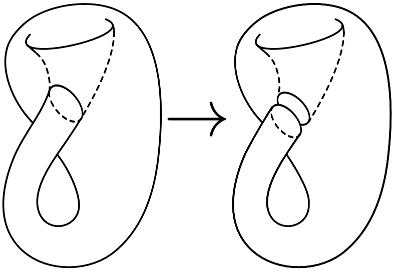
Example: the immersed Klein bottle

A Klein bottle immersed in $\mathbb{R}^3$ has an inside and an outside



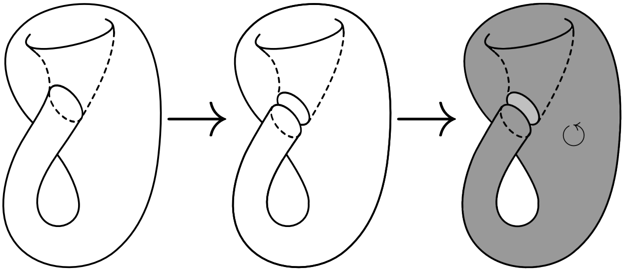
Example: the immersed Klein bottle

A Klein bottle immersed in $\mathbb{R}^3$ has an inside and an outside



Example: the immersed Klein bottle

A Klein bottle immersed in $\mathbb{R}^3$ has an inside and an outside



so it is orientable!

Results in low codimension

Proposition

Every $(n-1)$*–cycle in* $\mathbb{R}^n$ *is orientable, i.e.,* $\mathrm{NO}(A) = \mathrm{mass}(A)$.

Results in low codimension

Proposition

Every $(n-1)$*–cycle in* $\mathbb{R}^n$ *is orientable, i.e.,* $\mathrm{NO}(A) = \mathrm{mass}(A)$*.*

Corollary (Federer)

If T *is an integral* $(n-2)$*–cycle in* $\mathbb{R}^n$*, then* $\mathrm{FV}(2T) = 2\,\mathrm{FV}(T)$*.*

Results in low codimension

Proposition

Every $(n-1)$*–cycle in* $\mathbb{R}^n$ *is orientable, i.e.,* $\mathrm{NO}(A) = \mathrm{mass}(A)$.

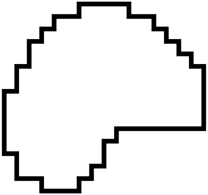
Corollary (Federer)

If T *is an integral* $(n-2)$*–cycle in* $\mathbb{R}^n$*, then* $\mathrm{FV}(2T) = 2\,\mathrm{FV}(T)$.

What about higher codimensions?

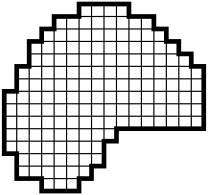
A simple argument in high codimension

Let A be a mod-2 cellular d-cycle of mass V



A simple argument in high codimension

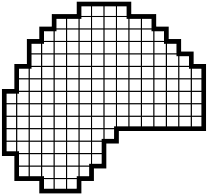
Let A be a mod-2 cellular d-cycle of mass V



- Fill A with a mod-2 chain F

A simple argument in high codimension

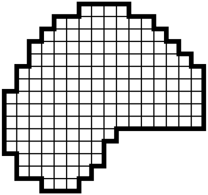
Let A be a mod-2 cellular d-cycle of mass V



- Fill A with a mod-2 chain F
- F is a sum of $V^{(d+1)/d}$ cubes, each with side length ~ 1

A simple argument in high codimension

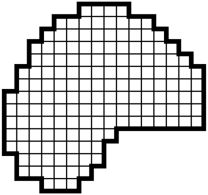
Let A be a mod-2 cellular d-cycle of mass V



- Fill A with a mod-2 chain F
- F is a sum of $V^{(d+1)/d}$ cubes, each with side length ~ 1
- Orient the cubes at random to get $F_{\mathbb{Z}}$

A simple argument in high codimension

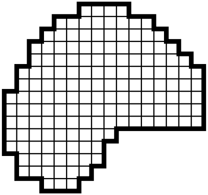
Let A be a mod-2 cellular d-cycle of mass V



- Fill A with a mod-2 chain F
- F is a sum of $V^{(d+1)/d}$ cubes, each with side length ~ 1
- Orient the cubes at random to get $F_{\mathbb{Z}}$
- $\partial F_{\mathbb{Z}}$ is a pseudo-orientation

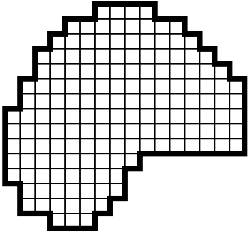
A simple argument in high codimension

Let A be a mod-2 cellular d-cycle of mass V



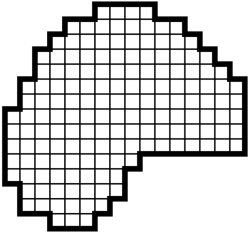
- Fill A with a mod-2 chain F
- F is a sum of $V^{(d+1)/d}$ cubes, each with side length ~ 1
- Orient the cubes at random to get $F_{\mathbb{Z}}$
- $\partial F_{\mathbb{Z}}$ is a pseudo-orientation
- $\mathrm{NO}(A) \lesssim \operatorname{mass} \partial F_{\mathbb{Z}} \sim V^{(d+1)/d}$

Bigger cubes

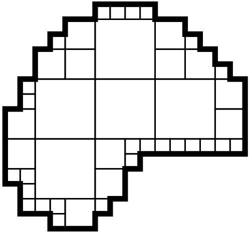


Total boundary: $V^{(d+1)/d}$

Bigger cubes

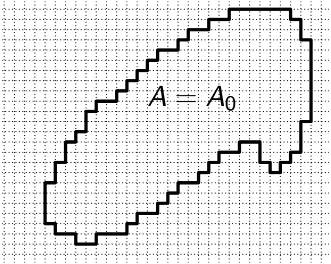


Total boundary: $V^{(d+1)/d}$

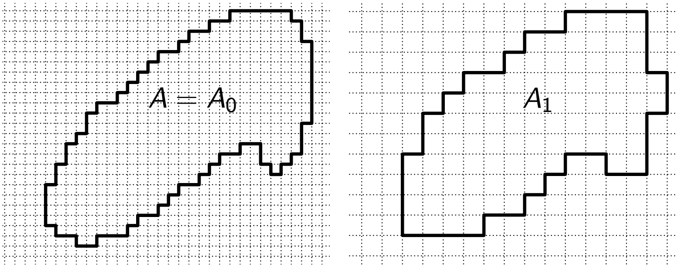


Total boundary: much less

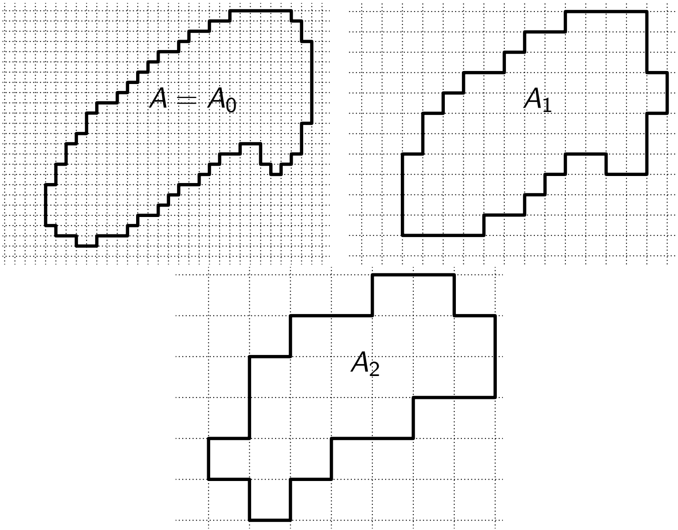
Filling through approximations



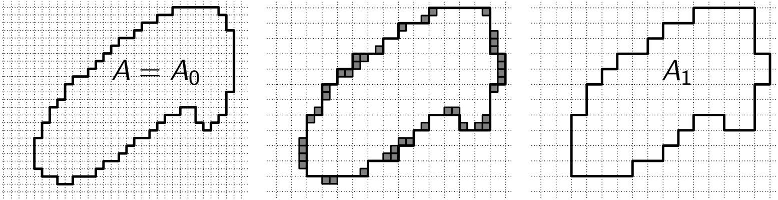
Filling through approximations



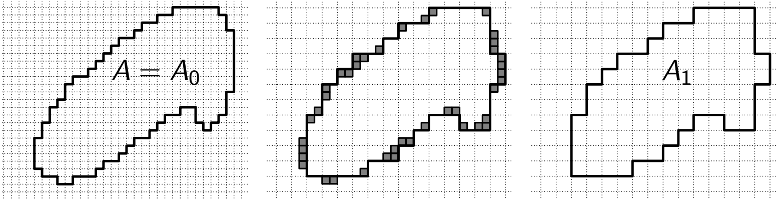
Filling through approximations



Filling through approximations

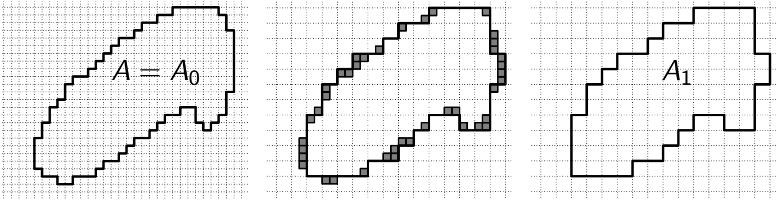


Filling through approximations

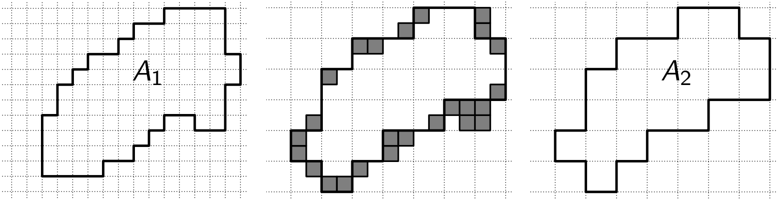


$\sim V$ squares each with perimeter ~ 1

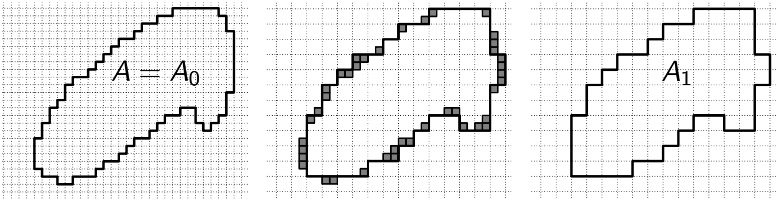
Filling through approximations



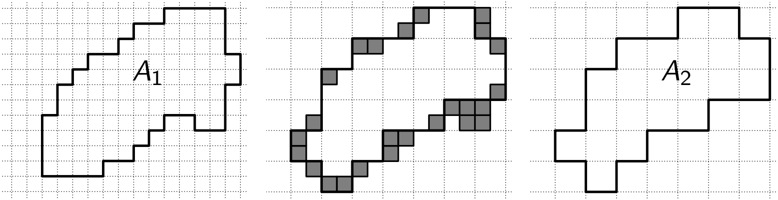
$\sim V$ squares each with perimeter ~ 1



Filling through approximations



$\sim V$ squares each with perimeter ~ 1



$\sim V/2$ squares each with perimeter ~ 2

Filling through approximations

Sketch:

- Approximate A at $\sim \log V$ scales, then connect the approximations.
- We use cubes with total boundary $\sim V$ at each scale.
- Since there are $\sim \log V$ scales, we conclude:

Proposition (Guth-Y.)

If A is a cellular mod-2 cycle with volume V, then it has a pseudo-orientation P such that $\operatorname{mass} P \lesssim V \log V$.

Filling through approximations

Sketch:

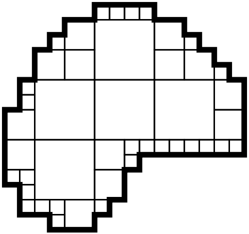
- Approximate A at $\sim \log V$ scales, then connect the approximations.
- We use cubes with total boundary $\sim V$ at each scale.
- Since there are $\sim \log V$ scales, we conclude:

Proposition (Guth-Y.)

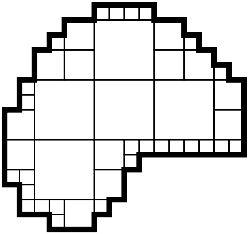
If A is a cellular mod-2 cycle with volume V, then it has a pseudo-orientation P such that $\operatorname{mass} P \lesssim V \log V$.

How do we get rid of the log factor?

Getting rid of the log factor

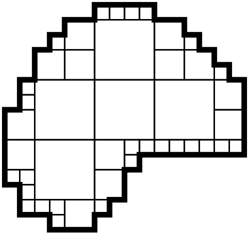


Getting rid of the log factor



- Choosing orientations randomly is wasteful when A is close to a plane

Getting rid of the log factor



- Choosing orientations randomly is wasteful when A is close to a plane
- But what if A is never close to a plane?

Dealing with complexity

How do we prove the proposition for sets that are close to fractals?

Dealing with complexity

How do we prove the proposition for sets that are close to fractals?

- Show that adding topological complexity adds extra area

Dealing with complexity

How do we prove the proposition for sets that are close to fractals?

- Show that adding topological complexity adds extra area
- Prove the theorem when A has “low complexity”

Uniform rectifiability

Definition (David-Semmes)

A set $E \subset \mathbb{R}^k$ is uniformly rectifiable if and only if E has a corona decomposition. (Roughly, for all but a few balls B, the intersection $B \cap E$ is close to the graph of a Lipschitz function with small Lipschitz constant.)

Sketch of proof

Proposition

Every mod-2 cellular d-cycle A can be written as a sum

$$A = \sum_i A_i$$

of mod-2 cellular d-cycles with uniformly rectifiable support such that

$$\sum \operatorname{mass} A_i \leq C \operatorname{mass} A.$$

Sketch of proof

Proposition

Every mod-2 cellular d-cycle A can be written as a sum

$$A = \sum_i A_i$$

of mod-2 cellular d-cycles with uniformly rectifiable support such that

$$\sum \operatorname{mass} A_i \leq C \operatorname{mass} A.$$

Proposition

Any mod-2 cellular d-cycle A with uniformly rectifiable support has a pseudo-orientation P with

$$\operatorname{mass} P \leq C \operatorname{mass} A.$$

Open questions

- Is $\mathrm{FV}(T) \geq \mathrm{FV}(T)$?

Open questions

- Is $\mathrm{FV}(T) \geq \mathrm{FV}(T)$?
- More generally,

$$\frac{\mathrm{FV}(kT)}{k} \geq c_k \, \mathrm{FV}(T).$$

  Can the c_k be chosen uniformly?

Open questions

- Is $\mathrm{FV}(T) \geq \mathrm{FV}(T)$?
- More generally,
$$\frac{\mathrm{FV}(kT)}{k} \geq c_k\,\mathrm{FV}(T).$$
Can the c_k be chosen uniformly?
- What does this tell us about the geometry of surfaces embedded in $\mathbb{R}^n$ by a bilipschitz map?