Solutions to Problem Set 4

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1. Let $Y$ be a compact subset of $X$ and let $U$ be an open set containing $Y$. For every $y \in Y$, there is an $r(y)$ such that $B_{r(y)}(y) \subset U$; let $A_y = B_{r(y)/2}(y)$. These sets form an open cover of $Y$, so there are finitely many points $y_1, \ldots, y_n$ such that $Y \subset \bigcup_i A_{y_i}$. Let $\epsilon = \min_i r(y_i)/2$ and suppose that $y \in Y$. We claim that $B_\epsilon(y) \subset U$. Suppose $z \in B_\epsilon(y)$. Let $i$ be such that $y \in A_{y_i}$. Then

$$d(y_i, z) \leq d(y_i, y) + d(y, z) < r(y_i)/2 + \epsilon < r(y_i),$$

so $z \in B_{r(y_i)}(y_i) \subset U$.

The set $A \times B$ is compact and the function $d$ is continuous, so by the Extreme Value Theorem, the restriction of $d$ to $A \times B$ achieves a minimum at a point $(a, b)$ such that $d(a, b) = d(A, B)$.

Oops, this turns out to be false – consider the interval $[-1, 1]$ with equivalence relation $x \sim y$ if $|x| = |y|$ and $|x| < 1$. The points $-1, 0, 1$ are in their own equivalence classes, and each other point is equivalent to its negative; we denote the equivalence class of $x$ by $[x]$ and let $q: X \to X/\sim$ be the quotient map $q(x) = [x]$.

Suppose that $U \subset X/\sim$ is an open set containing $[1]$. Then $q^{-1}(U)$ is an open set containing $1$; it thus contains an interval of the form $(1 - \epsilon, 1]$, so $q((1 - \epsilon, 1]) \subset U$. Likewise, if $V$ is a neighborhood of $[-1]$, then there is an $\epsilon' > 0$ such that $q([-1, -1+\epsilon'])$. These two sets overlap, so $X/\sim$ is not Hausdorff and thus not metrizable.

2. In class, we showed that if $X$ is Hausdorff, $Y$ is a compact subset of $X$, and $x \in X \setminus Y$, then there is a neighborhood $U_x$ of $Y$ such that $x \notin U_x$. Indeed, our proof showed that there is a neighborhood $V_x$ of $x$ such that $U_x \cap V_x \neq \emptyset$.

The sets $\{V_x\}_{x \in Z}$ form an open cover of $Z$. Let $z_1, \ldots, z_k$ be a finite subset such that $Z \subset \bigcup_i V_{z_i}$. Define $V = \bigcup_i V_{z_i}$ and $U = \bigcap_i U_{z_i}$. If $x \in V$, then $x \in V_{z_i}$ for some $i$, so $x \notin U_{z_i}$ and thus $x \notin U$. Therefore, $U \cap V = \emptyset$, as desired.

3. Since $X$ is compact, it is totally bounded, and for any $\epsilon$, there is a cover of $X$ by finitely many $\epsilon$-balls. For each $i > 0$, let $\{B_{2^{-i}}(x_{i,j})\}_{j=1}^{n_i}$ be a finite cover of $X$.

We will define a sequence of maps $F_i$ that converge uniformly to a map $F: S_2 \to X$. If $a = (a_1)_{i=1}^\infty \in S_2$, then $F_i(a)$ will depend on the first $N_i = \sum_{j=1}^i n_i$ terms of $a$. Let $p_i: S_2 \to \{0, 1\}^{n_i}$ be the map

$$p_i(a) = (a_{N_i-1+1}, \ldots, a_{N_i}).$$

Let $\sigma_i$ be a surjective map from $\{0, 1\}^{n_i}$ to $\{1, \ldots, n_i\}$ and let

$$P_i(a) = x_{i, \sigma(p_i(a))}.$$
Note that $P_i$ depends only on the first $i$ terms of $a$, so it is continuous.

Let $F_1 : S_2 \to X$ be the map $F_1 = P_i$. For each $i > 1$, let $F_i : S_2 \to X$ be the map

$$F_i(a) = \begin{cases} P_i(a) & \text{if } \bigcap_{j=1}^{i} B_{2^{-j}}(P_j(a)) \neq \emptyset \\ F_{i-1}(a) & \text{otherwise.} \end{cases} \quad (1)$$

Each such map is continuous, and for each $i$, $a$, either $F_i(a) = F_{i-1}(a)$ or the intersection of $B_{2^{-i}}(F_i(a))$ and $B_{2^{-i+1}}(F_{i-1}(a))$ is nonempty. In the first case, $d(F_i(a), F_{i-1}(a)) = 0$. In the second, $d(F_i(a), F_{i-1}(a)) \leq 2^{-i+2}$. It follows that the sequences $F_i(a)$, as $i$ goes to infinity, are uniformly Cauchy. Since a compact metric space is sequentially compact, these sequences converge uniformly, and we can define

$$F(a) = \lim_{i \to \infty} F_i(a).$$

This is a uniform limit of continuous functions, so it is continuous.

Finally, we claim that $F$ is surjective. If $x \in X$, then there is a sequence of $k_i$’s such that $x \in B_{2^{-i}}(x_{1,k_1})$ for all $i$ and there is a $a$ such that $P_i(a) = x_{i,k_i}$ for all $i$. It follows that

$$x \in \bigcap_{i=1}^{\infty} B_{2^{-i}}(P_i(a)),$$

and thus that $F_i(a) = x_{1,k_1}$. Then $\lim_i F_i(a) = x$, as desired.

4. Suppose that $K \subset X \times Y$ is a closed set. We claim that $p(K)$ is closed. Suppose $x \in X \setminus p(K)$. Then $p^{-1}(x) = x \times Y$ is a compact subset of $X \times Y$ that is disjoint from $K$.

To separate $p^{-1}(x)$ from $K$, we use an argument from class (the Tube Lemma). For every point $(x, y) \in p^{-1}(x)$, there is a basis element $A_y \times B_y$ such that $A_y$ is a neighborhood of $x$, $B_y$ is a neighborhood of $y$, and $A_y \times B_y$ is disjoint from $K$. There are finitely many such neighborhoods $A_{y_i} \times B_{y_i}$ that cover $p^{-1}(x)$, and we let $A = \bigcap_i A_{y_i} \subset X$. Then $A$ is a neighborhood of $x$ such that $A \times Y \cap K = \emptyset$, so $A \cap p(K) = \emptyset$.

For the counterexample, let $X = Y = \mathbb{R}$. Then $p$ is not a closed map; if $K = \{(x, y) \mid xy = 1\}$, then $K$ is closed, but $p(K) = \mathbb{R} \setminus \{0\}$ is not.