Solutions to Problem Set 2

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1. Suppose that $X$ is Hausdorff. We claim that $D^c = X \times X \setminus D$ is open. Let $(x, y) \in D^c$. Then $x \neq y$. We claim that there is some basis element such that $x \in U \subset D^c$. Since $X$ is Hausdorff, there are open sets $A, B \subset X$ such that $x \in A$, $y \in B$, and $A \cap B = \emptyset$. Then $A \times B \subset D^c$ and $(x, y) \in A \times B$.

Now suppose that $D^c$ is open. Let $x, y \in X$ be two distinct points. Then $p = (x, y) \in D^c$, so there is some basis element $A \times B \subset X \times X$ such that $p \in A \times B \subset D^c$ and such that $A$ and $B$ are open. It follows that $x \in A$ and $y \in B$ and $A \cap B = \emptyset$. This is true for any two distinct $x$ and $y$, so $X$ is Hausdorff.

2. Consider the order topology on $\mathbb{N} \cup \{\infty\}$. (The subspace topology on $0, 1/2, 2/3, 3/4, \ldots, 1$ gives rise to the same topology.) In this topology, any subset of $\mathbb{N}$ is open, because every one-point subset of $\mathbb{N}$ is an open interval. On the other hand, a set $U$ containing $\infty$ is open if and only if it contains some open interval containing $\infty$. That is, if and only if $(N, \infty) \subset U$ for some $N \in \mathbb{N}$.

Suppose that $f: \mathbb{N} \cup \{\infty\} \to X$ is a continuous map. If $U \subset X$ is an open set containing $f(\infty)$, then $f^{-1}(U)$ is an open set containing $\infty$, so there is some $N \in \mathbb{N}$ such that $(N, \infty) \subset f^{-1}(U)$. That is, $f(n) \in U$ for all $n > N$. Since this is true for every open subset $U$, it follows that $f(n)$ converges to $f(\infty)$.

Conversely, suppose that $a_i$ is a sequence converging to $L$ and that $f: \mathbb{N} \cup \{\infty\} \to X$ is the map such that $f(i) = x_i$, $f(\infty) = L$. Let $U \subset X$ be an open set. If $L \notin U$, then $f^{-1}(U) \subset \mathbb{N}$, so $f^{-1}(U)$ is open. On the other hand, if $L \in U$, then there is an $N \in \mathbb{N}$ such that $f(n) \in U$ for all $n > N$, so $(N, \infty) \subset f^{-1}(U)$, and $f^{-1}(U)$ is open.

3. If $F$ is closed, then $F \cap U_\alpha$ is closed with respect to the subspace topology by definition.

Conversely, suppose that each $F \cap U_\alpha$ is closed. Let $G_\alpha = U_\alpha \setminus F$. This set is open in the subspace topology on $U_\alpha$, so $G_\alpha = V \cap U_\alpha$ for some open set $V \subset X$ with $U_\alpha$. Therefore, $G_\alpha$ is open in $X$, and $\bigcup_\alpha G_\alpha = X \setminus F$. This is a union of open sets in $X$, so it is open, and thus $F$ is closed.

4. Let $x, y \in X$ be such that $x \neq y$ and let $f: X \to \mathbb{R}$ be such that $f(x) \neq f(y)$. Let $\epsilon \in (0, |f(x) - f(y)|/2)$, so that the intervals $A = (f(x) - \epsilon, f(x) + \epsilon)$ and $B = (f(y) - \epsilon, f(y) + \epsilon)$ are disjoint. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint open sets such that $x \in f^{-1}(A)$ and $y \in f^{-1}(B)$. Therefore, $X$ is Hausdorff.

5. The pasting lemma tells us that if $X = K_1 \cup \cdots \cup K_n$ and the $K_i$ are closed sets, then $f: X \to Y$ is continuous if $f|_{K_i}: K_i \to Y$ is continuous for all $i$. Is there an analogue of the pasting lemma when there are countably many $K_i$?
No. For example, consider \( f : \mathbb{R} \to \mathbb{R} \),
\[
    f(x) = \begin{cases} 
        0 & \text{if } x < 0 \\
        1 & \text{if } x \geq 0
    \end{cases}
\]
This is discontinuous, but if \( K_0 = [0, \infty) \), \( K_1 = (-\infty, -1] \), and \( K_i = [-\frac{1}{i-1}, -\frac{1}{i}] \) for \( i > 1 \),
then each \( K_i \) is closed, \( \bigcup K_i = \mathbb{R} \), and \( f|_{K_i} \) is continuous for all \( i \).

6. In problem set 1, we considered a metric topology on the set \( S_2 \) of 0-1 sequences. Suppose that for all \( i \), \( \{a^i_n\}_{n=1}^{\infty} \) is a 0-1 sequence. Show that
\[
    \lim_{i \to \infty} \{a^i_n\}_{n=1}^{\infty} = \{b_n\}_{n=1}^{\infty}
\]
if and only if \( \lim_{i \to \infty} a^i_n = b_n \) for every \( n \).
First, suppose that \( \lim_{i \to \infty} a^i_n = b_n \) for every \( n \). This implies that for every \( n \), there is a \( k_n \)
such that for all \( i > k_n \) we have \( a^i_n = b_n \). Consequently, if \( i > \max\{k_n \mid n = 1, \ldots, N\} \), then
\[
    d(\{a^i_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}) < N^{-1}.
\]
It follows that
\[
    \lim_{i \to \infty} \{a^i_n\}_{n=1}^{\infty} = \{b_n\}_{n=1}^{\infty}.
\]
Conversely, suppose that
\[
    \lim_{i \to \infty} \{a^i_n\}_{n=1}^{\infty} = \{b_n\}_{n=1}^{\infty}
\]
and let \( N > 0 \). There is some \( K \) such that if \( k > K \), then
\[
    d(\{a^k_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}) < N^{-1}.
\]
Therefore, \( a^k_N = b_N \) for all \( k > K \), and \( \lim_{i \to \infty} a^i_N = b_N \).