Solutions to Problem Set 1

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September 19, 2015

1. Recall that the rational numbers are dense in the real numbers, so for every \( a, b \in \mathbb{R} \) with \( a < b \), there is a \( q \in \mathbb{Q} \cap (a, b) \). In particular, for any \( x \in \mathbb{R} \), there are \( a, b \in \mathbb{Q} \) such that \( x \in (a, b) \).

If \((a, b), (c, d) \in B_\mathbb{Q}\) intersect, then \((a, b) \cap (c, d)\) is an interval with rational endpoints, so it is also a basis element. It follows that \(B_\mathbb{Q}\) is a basis.

Let \( T_\mathbb{Q}\) be the topology generated by \(B_\mathbb{Q}\) and let \( T\) be the standard topology. The basis \(B_\mathbb{Q}\) is a subset of the standard basis for the standard topology, so \( T_\mathbb{Q} \subset T\).

On the other hand, if \((a, b) \in B_\mathbb{Q}\) and \( x \in (a, b) \), then there are rational numbers \( a' \in (a, x) \) and \( b' \in (x, b) \). Then \((a', b') \in B_\mathbb{Q}\) and \((a', b') \subset (a, b) \). It follows that \((a, b)\) is open in \(T_\mathbb{Q}\), so \( T \subset T_\mathbb{Q}\).

2. (Solution by Max Fishelson) The basis of the lexicographic topology consists of lexicographic intervals \(((a, b), (c, d))\): the set of all points of the form \((x, y)\) with \(a < x < c\) or \(x = a, y > b\) or \(x = c, y < d\). The basis of \(\mathbb{R}_d\) is the set of all points in \(\mathbb{R}\), and the basis of the standard topology in \(\mathbb{R}\) is the set of all open intervals in \(\mathbb{R}\). So, the basis for \(\mathbb{R}_d \times \mathbb{R}\) would consist of sets of points satisfying the following: for reals \(a, b, c\), all points of the form \((x, y)\) with \(x = a, b < y < c\). Let us call these basis elements “vertical intervals” and denote them as \(v(a, b, c)\). We now want to show that the basis of lexicographic intervals and the basis of vertical intervals generate the same topology.

(a) We want to show \(\forall\) vertical intervals \(v(a, b, c)\), \(\forall x \in v(a, b, c)\), \(\exists\) lexicographic interval \(((m, n)(p, q))\) such that \(x \in ((m, n)(p, q)) \subseteq v(a, b, c)\).

Then \(m = a, n = b, p = a, q = c\) satisfies the conditions.

(b) We want to show \(\forall\) lexicographic intervals \(((m, n)(p, q))\), \(\forall (x, y) \in ((m, n)(p, q))\), \(\exists\) vertical interval \(v(a, b, c)\) such that \((x, y) \in v(a, b, c) \subseteq ((m, n)(p, q))\).

If \(x = m, a = m, b = n, c = y + 1\) satisfies the conditions. If \(x = p, a = p, b = y − 1, c = q\) satisfies the conditions. Otherwise \(a = x, b = y − 1, c = y + 1\) satisfies the conditions.

3. It is clear that if \(a, b \in S_2\), then \(d(a, b) = d(b, a)\) and that \(d(a, b) = 0\) if and only if \(a = 0\). It remains to prove that \(S_2\) satisfies the triangle inequality and the ultrametric inequality.

Suppose that \(a, b, c \in S_2\). Let \(j\) be a positive integer. If \(d(a, b) \leq 1/j\), then \(a_i = b_i\) for \(i < j\). Likewise, if \(d(b, c) \leq 1/j\), then \(b_i = c_i\) for \(i < j\). It follows that if \(d(a, b) \leq 1/j\) and \(d(b, c) \leq 1/j\), then \(a_i = b_i = c_i\) for \(i < j\), so \(d(a, c) \leq 1/j\). Thus, for all \(a, b, c \in S_2\),

\[
d(a, c) \leq \max\{d(a, b), d(b, c)\}.
\]
Finally,  

$$\max\{d(a,b), d(b,c)\} \leq d(a,b) + d(b,c),$$

so the ultrametric inequality implies the triangle inequality.

4. (a) Open balls in an ultrametric space have many centers. That is, for all $x \in X$ and $r > 0$, if $y \in B_r(x)$, then $B_r(x) = B_r(y)$.

Suppose that $x \in X$, $r > 0$ are such that $y \in B_r(x)$. Then $d(x,y) < r$. If $z \in B_r(x)$, then $d(x,z) < r$, so 

$$d(y,z) < \max\{d(x,z), d(x,y)\} < r,$$

and $z \in B_r(y)$. Thus $B_r(x) \subset B_r(y)$. Conversely, $x \in B_r(y)$, so $B_r(y) \subset B_r(x)$, and the two balls are equal.

(b) Let $B = B_r(x)$ and $B' = B_{r'}(y)$ and suppose that $B \cap B' \neq \emptyset$. Let $z \in B \cap B'$. By the previous question, $B = B_r(z)$ and $B' = B_{r'}(z)$, so we have $B \subset B'$ or $B' \subset B$ depending on whether $r \leq r'$ or $r' \leq r$.

(c) Suppose that $\{y_i\}_{i=1}^{\infty}$ is a sequence in $B_r(x)$ such that $y_i$ converges to $y$. Let $B' = B_r(y)$. Then $y_i \in B'$ for all sufficiently large $i$, so $B'$ and $B_r(x)$ intersect. By our previous work, $B' = B_r(x)$, so in particular, $y \in B_r(x)$. 

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