Last time we talked about geometry of constant curvature, put some flat, hyperbolic, structures on surfaces. Today I'd like to describe the set of geometric structures on surfaces.

We would like to define something like $M(\Sigma) = \text{set of hyperbolic surfaces homeo to } \Sigma$, but it turns out that the better way to define moduli space is to define it as a quotient of Teichmüller space $\mathcal{T}(\Sigma) = \text{set of geometric structures on } \Sigma$.

**Def.:** A geometric structure on $\Sigma$ consists of a manifold $X \cong \Sigma$ with a hyperbolic metric w/ geodesic boundary or a flat, a real metric w/ geodesic boundary, along with a homeomorphism $\phi : \Sigma \to X$ (a marking).

We say two geometric structures are equivalent if there is a homeomorphism $\phi : X' \to X$ s.t. $\phi \circ \phi'$ is isotopy.

(Ref: If $X, Y$ are compact surfaces or surfaces w/ finitely many punctures, then $\text{Homeo}(X, Y)$ is $\text{Diffeo}(X, Y)$.)

The difference between these is the marking, not just the metric, but the way that the metric sits on $\Sigma$. Let's see an example.

$\Sigma = S^1 \times S^1$.

$M(\Sigma) = \{\text{all flat tori of area } 1\}$. $\cong \text{flat tori } / \text{isometry} \cong \text{flat tori } / \text{scaling}$.

Flat torus $\Sigma$.

$\mathbb{M}(\Sigma)$ acts freely on $\mathbb{R}^2$.

$M(\Sigma)$ acts freely on $\mathbb{R}^2$ by translations.

$\mathbb{R}^2 / \mathbb{M} \cong \mathbb{R}^2$.

$\mathbb{M}$ is a lattice in $\mathbb{R}^2$.

So $M(\Sigma)$ is the lattices in $\mathbb{R}^2$ up to scaling and rotation.

Where $\mathbb{M}$ is generated by two translations (lattice).

This is not unique:

$\mathbb{R}^2 / \mathbb{P}$ (where $\mathbb{P}$ is $\mathbb{M}$, a rotation).

So $M(\Sigma)$ is lattices in $\mathbb{R}^2$ up to scaling and rotation.
Parameterizing $M(3)$: Let $T : M(3)$, let $v$ be the shortest vector.

We can scale and rotate so $v = (1,0)$.

Let $w \in \mathbb{R}^2$ and $p(w) = \langle w, (1,0) \rangle$. We can scale and rotate a lattice so that one generator is $(1,0)$, and define $p : \mathbb{R}^2 \to M(3)$.

Technically, $p : \mathbb{R} \times \mathbb{R}^2 \to M(3)$, to avoid the $x$-axis.

But in general, each lattice is the image of infinitely many $v$ - let's find a fundamental domain - to do this, we need to associate a region over to $C = \mathbb{R}$, rotate/scale so that $(1,0)$ is a generator of minimal length.

Where is the generator? A lattice is in a standard way - if $v \in \mathbb{R}^2$, let $v$ be the generator of minimal length, $\vec{v}$ a generator of minimal length not in $\mathbb{R}^2$.

Rotate and scale so $w = (1,0)$, where $w$ - above $x$-axis (otherwise, replace $w$ by $-w$).

Outside unit circle

$x$-coordinate between $\frac{1}{2}, \frac{3}{2}$

Some point: Every lattice has at most one precision in int $(S)$.

Draw identity on below.

But there isn't the only local domain!

$p(x,y) = p(x + 1, y)$.

If we view $S \subset H$, then $H$ is tiled by copies of $S$ - modular tiling.

In fact, $\mathcal{T}(S3) = H$: each geometric structure $p(x,y)$.

These are isometric, but the isometry doesn't commute with the marking.

So $\mathcal{T}(S)$ maps upper half-plane $\to \mathcal{T}(S)$.
Backwards? Suppose flat torus:

\[ \phi \]

\[ \pi_1(S) = \langle \alpha, \beta \rangle \]

1. Straighten to geodesics.
2. Lift to universal cover.

\[ \tilde{\pi}_1(S) = \langle \phi_*(\alpha), \phi_*(\beta) \rangle \]

\[ \tilde{\pi}_1(x) = \langle \phi_*(\alpha), \phi(\beta) \rangle \]

\[ \pi_1(S) = \langle \alpha, \beta \rangle \]

\[ x \equiv \alpha \]

The mapping class group of \( S \) is defined as:

\[ \text{Mod}(S) = \pi_0(\text{Homeo}^+(S, \partial S)) \]

\[ \text{Homeo}^+(S, \partial S) = \text{orientation-preserving self-homeos that fix the boundary pointwise}. \]

Note: if \( S \) is compact or compact with finite punctures,

\[ \text{Mod}(S) = \pi_0(\text{Diff}^+(S, \partial S)) \]

\[ = \text{Homeo}^+(S, \partial S) / \text{homeos} \]

Ex: \( S = D^2 \Rightarrow \text{Mod}(S) = \mathbb{Z} \)

(Alexander trick)

\[ S = \square \]

\[ \chi(\text{Mod}(S)) \]

\[ f : \text{Mod}(S) \rightarrow \mathbb{Z} \]

\[ f \xrightarrow{\text{winding number of}} \mathbb{Z} \]

Prop: This is an isomorphism. Clearly surjective:

\[ f_n \circ \theta = (r \theta + 2 \pi n (r-1)) \]

Where does this group action come from?

The group action changes the markings: if \( S \) is a surface, we define the mapping class group of \( S \) by
If $\Theta(t) = n$, let $\gamma_n = \text{quasi-pulling a } \gamma$.

Then, there is an isomorphism taking $f(g)$ to $g^n$.

Then the Alexander trick applies.

**$S = T^2$:** If $f \in \text{Mod}(S^2)$, then $f$ induces a map

$$f^* : \pi_1(T^2) \to \pi_1(T^2)$$

So

$$\sigma : \text{Mod}(T^2) \to \text{GL}_2(\mathbb{Z}), \quad \sigma(f) = f^*$$

In fact, since $f$ is orientation-preserving, it preserves the oriented intersection number of a pair of curves $x \in \text{SL}_2(\mathbb{Z})$.

**$\sigma$ is surjective:** Suppose $M \in \text{SL}_2(\mathbb{Z})$ then $M : \mathbb{R}^2 \to \mathbb{R}^2$ descends to a map $T^2 \to T^2$ s.t. $\sigma(M) = M$.

**$\sigma$ is injective:** If $f : T^2 \to T^2$, then we can lift $f$ to a map $f' : \mathbb{R}^2 \to \mathbb{R}^2$.

**This is homotopic to all**.

Say $\sigma(f) = M$. Then

$$h(x,t) = f(x) + (1-t)Mx$$

is a homotopy from $f$ to $M$ and if $z \in \mathbb{Z}^2$, then

$$h(x + z, t) = (1-t)f(x + z) + (1-t)Mx + (1-t)Mz$$

So $h$ descends to a homotopy $\tilde{h}$ from $f$ to $M$.

And this acts on $\text{Tor}(S)$.

And $\text{Mod}(S)$ acts on $\text{Tor}(S)$: if $f \in \text{Mod}(S)$, then let $f \circ \phi = \phi \circ f^{-1}$.

**Check:** This is well-defined. **Fixed points:**

In our example,

$$
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\quad \text{acts as} \quad (x,y) \mapsto (x+y, y)
$$

$$
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\quad \text{acts as} \quad z \mapsto -\frac{1}{z}
$$

$$
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\quad \text{is an isometry, so it acts as identity.}
$$

(Indeed, $\text{Thm. 3}(M) \equiv \text{PSL}_2(\mathbb{R}) \equiv \frac{\text{SL}_2(\mathbb{R})}{\pm 1}$.)