"Def": The Brownian excursion is a Brownian motion conditioned so that \( f(0) = f(1) = 0 \) and \( f \geq 0 \) on \([0, 1]\).

There are a couple of ways to fix this. Here is a version due to Levey:
let \( f \) be a Brownian motion, conditioned so \( f(1) = 0 \)
\[
\begin{align*}
& t_+ = \sup \{ t : f(t) = 0, t < 1 \} \\
& t^- = \inf \{ t : f(t) = 0, t > 1 \}
\end{align*}
\]
Let \( g \) be \( f \) rescaled so that \( \int t^{-1} g(t) \) goes to \([0, 1] \)
and with its height stretched by \( t_+^{-1} \). Then \( e \) is a Brownian excursion, rescaled.

Thm: Random 2n-step excursions \( \rightarrow \) Brownian excursion and so
Thm (Aldous): rescaled random trees \( \rightarrow \) trees
If \( T_n \) is the n-vertex uniform random tree
and \( e \) is the Brownian excursion, then

\[
(T_n, \ln T_n) \rightarrow T_e
\]
Furthermore, this is universal - different models of uniform random tree have the same limit up to a scaling.

Random spheres and planes:
Q) Are there finitely many ways to triangulate a sphere with \( n \) vertices?
- What does a random one look like? (Connections to quantum gravity)
- Like the trees, we'll have two approaches - local, based on counting, one global, based on a bijection.

Local - the Uniform Infinite Planar Triangulation (Angel, Schramm)
Last week, we discussed the shape of the top \( K \) levels of a random tree - this week, balls of radius \( r \) in a random triangulation.

Let \( T \) be the set of connected planar rooted triangulations
(2d complexes embedded in the plane with accumulation points of vertices) We say \( T_i \rightarrow T \) if \( T_i \) agrees with \( T \) on larger and larger balls around the root. Then:

Thm (Angel, Schramm): The uniform distribution \( T_n \) on n-vertex triangulations of \( S^2 \) converges weakly to a prob measure on \( T \) as the number of vertices of triangulations of the plane. Such triangulations are called triangulations of the plane.
What does this mean? This is really two claims.

1. Tightness - the balls around the root stay controlled as \( n \to \infty \).

   If \( X \in T \), then

   \[ T(X) = \{ Y \in T \mid \exists \text{ contains } X \text{ as a rooted triangulation} \} \]

   There is a countable set \( X_1, X_2 \ldots \)

   of possible balls of radius \( r \), so the most have \( P \circ [T(X)] \to P \), and

   claim \( \sum P_i = 1 \).
A.s. planarity

Claim: A random triangulation is a.s. one-ended.

Sketch: Let $U_{k, l} \overset{d}{=} \{ \text{there is a rooted curve of length } k \text{ with } \geq l \text{ triangles on each side} \}$

Then $P(U_{k, l}) = \lim_{n \to \infty} P_n(U_{k, l})$.

Claim: $\forall k, \lim_{l \to \infty} P[U_{k, l}] = 0$
What does this mean? Ex: convergence of random graphs (in particular, tightness) to Poisson-Dirichlet process, and vertex degree approaches a fixed distribution. Ex: No curves that separate into two large pieces.

Sketch of sketch: Instead of building a tree on level at a time, we build a triangulation one annulus at a time-root, then a shell around root, etc.

What can happen?
- boundary components collapse,
- boundary components split,
- or grow/shrink.

And the probabilities are governed by the # of discs with a given boundary.

Suppose \( P \), \( A \) has \( k \) boundary components:

\[
T_1 \quad b_1 \\
\vdots \\
T_k \quad b_k
\]

Fit \( A \) into a sphere, we need \( k \) discs.

The total # of ways to complete \( A \) to a sphere, adding \( n \) vertices is

\[
\geq C_{b,a} \cdots C_{b,a_k}, \text{ where } C_{b,a} = \# \text{ of discs with } a \text{ boundary and } b \text{ interior vertices.}
\]

This # is known and

Thm: With prob \( \to 1 \) as \( n \to \infty \), there is an \( i \) s.t. \( a_i \to \infty \).

In fact, condition on \( i = 1 \), for \( j \neq 1 \)

\[
T_j \xrightarrow{d} \text{prob measure on } \mathbb{N}
\]

\[
T_j \xrightarrow{d} \text{prob measure on triangulation of a } b_j \text{-gon.}
\]

\[
T_i \xrightarrow{d} \text{prob measure on infinite triangulation of a } b_i \text{-gon.}
\]

In particular, if we condition on \( B_1 \), the triangulations and on which boundary component bounds the most vertices, the triangulations of the boundary are independent (in the limit).

We can describe the balls in terms of a multistate branching process.
Start with the root triangle. This has a 3-vertex boundary, and all the remaining vertices lie outside that boundary.

There are countably many annuli that can occur as the neighborhood of the root. We can construct a prob. dist. on them.

One of those contains almost all the remaining vertices, the other has only finitely many.

Continue building. All the non-infinite branches die off a.s., these are the tree triangulations.

So the limit has a tree structure.

Further, Angel analyzed this to get growth estimates:

\[ \limsup_{r \to \infty} \frac{18r}{r + \log r} \leq c \]

\[ \lim_{r \to \infty} \frac{18r}{r \log r} \rightarrow 17 = \infty \]

Similarly, up to polylogs, boundary of infinite component has
- lots of branching,
- suggests that diameter of \( T_n \) should be \( n^{\frac{1}{4}} \).

Rescaled random spheres: Q: What is \( \lim_{n \to \infty} (T_n, d_{T_n}) \)?

There's a beautiful bijection that describes certain spheres, due to Con, Vaugelas, and Scheffer.

Def: If \( T \) is a rooted plane tree whose vertices are labeled by positive integers, we say it is rescaled random trees.

Q: What is \( \lim_{n \to \infty} (T_n, d_{T_n}) \)?

Physicists believe - this limit exists - is a sphere with Hausdorff dimension 4.

Like before, there's a beautiful connection to Brownian motion - the limit is something. One called the Brownian map.

\[ d_2 < e^{v(x)}d_2 \]

This is based on a bijection to labeled trees, where \( h \) is the GFF.

Def: If \( T \) is a rooted plane tree whose vertices are labeled by positive integers, we say that \( T \) is well-labeled if...
The root is labeled $1$ and the labels of adjacent vertices differ by at most $1$.

**Theorem (V. Vaugelde-Schnetzer):**

Well-labeled $n$-edge trees $\rightarrow$ rooted quadrangulations with $n$ edges.

(Note: self-loops, multiple edges are allowed.)

**Example:**

- Why is this a quadrangulation?
  - There is one quad for each edge, with one of two types:

```
    x     x     x
  X x X x X
    x     x-1 x-1

  X-1        x-2
```

**Claim:** And this is planar $\Rightarrow$ sphere.

**Why is this invertible?** Given a quadrangulation, we can label the vertices by distance to origin, then construct. In each quad, we add one tree edge, depending on the labels. Claim: the result is a tree.

So, we can construct a random quadrangulation by:

1. Generate a random tree from a random walk.
2. Generate a random labeling well-labeling.
3. Translate so that the minimum label.

Typically, this is:

1. Construct a random tree.
2. Label the vertices starting at the root and adding $0$, $1$, $0$, or $1$ at each edge.
3. Translate the labels so that the minimum is $0$.

4. Construct a quadrangulation.

(The only problem is that the root.
5. Perhaps a root.

If we do this carefully, last step right we get the uniform distribution on rooted quadrangulations.
The Brownian map: There is a process called the Brownian snake that produces a CRT labeled by Brownian motion.

Random trees $\rightarrow$ Continuum random tree

With labeled path $\rightarrow$ path labeled by a Brownian motion.

This gives us two functions:

$c: [0,1] \rightarrow \mathbb{R}$ contour function

This has a continuous model:

random trees $\rightarrow$ continuum random tree

labelings $\rightarrow$ labeling on CRT

The Brownian snake is a random process that produces a CRT labeled so that every path from the root is labeled by Brownian motion.

Question: Can we use the Brownian snake to construct a random metric on $S^2$?

Proof:

There is a random metric space $(T, d)$ based on the Brownian snake such that $(T, d) \rightarrow (T, d)$ under the Gromov-Hausdorff topology. This is homeomorphic to $S^2$ and has Hausdorff dimension 4.

Conj: This should be correct from a metric calsphere

$dg^2 = e^{2\Phi(x)} dx^2$

on the sphere, where $\Phi(x)$ is the Gaussian Free Field on $S^2$. 
Problem: We don't know how to define this metric.

One strategy: each $T_n$ is conformally equivalent to $S^2$.

The uniform measure on the vertices should converge to a measure on $S^2$ — what is this measure?