Properties of random surfaces. Most of these were also properties of random regular graphs — some short curves, but not many. In a random regular graph, 1 vertex is linked to itself, but the ball around a point is probably a tree until that ball hits √N/2 vertices.

Today, a more 2-dimensional property...

Q: What's the shortest pants decomposition of a surface?

If $x_1, \ldots, x_3$ is a collection of disjoint geodesics in $S_g$,

define their lengths of $x_1, \ldots, x_3$ as

$$\text{total length} = \sum \ell(x_i)$$

If $\Sigma \times \mathbb{R}$ is a hyperbolic surface, with genus $g$, let

total pants length $\text{PL}(\Sigma) = \inf \left\{ \max \{ \ell(x_i) \} \right\}$

(a reeds of a hyperbolic manifold...)

Q: How do PL, TPL depend on genus?

(related to shape of $\Sigma$)

Upper bounds: We're going to need some called the Thick-Thin decomposition of a hyperbolic manifold.

Thm (Collar Theorem): If $\Sigma$ is a hyperbolic surface with geodesic boundary and if $x_1, \ldots, x_m$ are disjoint simple closed geodesics, then

the collar $C(x_i) = \left\{ p \in \Sigma \mid d(p, x_i) \leq w(x_i) \right\}$,

where $w(x_i) = \sinh^{-1} \left( \frac{1}{\sinh(\ell(x_i)/2)} \right)$

are pairwise disjoint, and

$$C(x_i) \approx \left[ -w(x_i), w(x_i) \right] \times S^1$$

$dg^2 = dp^2 + \lambda_i^2 \cosh^{-2} p \ dt^2$

If $\ell \leq 1$...

Proof: Complete $x_1, \ldots, x_m$ to a pants decomposition. Then it suffices to prove that each boundary component of a pair of pants has a half-collar...
Cor: $\exists \varepsilon_0 > 0$ s.t. if $\delta_1, \delta_2$ are segs and $l(\delta_i) < \varepsilon_0$, then $\delta_1$ and $\delta_2$ are disjoint.

Proof: Lift to universal cover $\tilde{X}$, so s.t. $w(\varepsilon) \geq 2\varepsilon_0$.

This segment has length $\geq 2w(\varepsilon) \geq 2\varepsilon_0$.

$\varepsilon$ here are $\leq 2g + b$ short closed geodesics. If $X$ is a closed surface

Then (Thick-thin decomposition): $\exists \varepsilon_0 > 0$ (in fact, $\varepsilon_0 = 2\operatorname{arcosh}(1)$).

So if $X$ is a closed surface and $l(\delta_i) < \varepsilon_0$, then $\delta_i$ are pairwise disjoint and $l(\delta_i) \leq \varepsilon_0$.

Proof: Suppose $\operatorname{inj}(X) \leq \varepsilon_0$. Then $X$ based at $x$ s.t. $\exists \delta \leq \varepsilon_0$, and $x$ a seg freely homotopic to $y$ s.t. $l(\delta) \leq \varepsilon_0$.

Then the seg is freely homotopic to some $\beta = \beta_i$, and $d(x, \beta) \leq \varepsilon_0$.

If $\varepsilon_0$ is small enough, then $d(x, \beta) \leq \varepsilon_0$, so $x \in C(\beta)$.

Let $X = X_{\geq \varepsilon} \cup X_{\leq \varepsilon}$, where $X_{\leq \varepsilon} = \{ x \in X \mid \operatorname{inj}(x) \leq \varepsilon_0 \}$.

If $X$ is closed and $e < 1$, then $X_{\leq \varepsilon}$ consists of disjoint cylinders (cylinders).

If $X$ is any hype surface, then $X_{\geq \varepsilon}$ consists of cylinders and caps.

Theorem (Benedetto, Ghilb): If $X$ is a closed hype surface of genus $g$, then $X_{\geq \varepsilon}$ has a pants decomposition of length $PL(X) \leq 9g^2$, $TPL \leq 9^2$.

Proof: Let $\beta_1, \ldots, \beta_n$ be the curves in $X$ with $l(\beta_i) < \varepsilon_0$.

These, plus a decom of $X_{\geq \varepsilon}$, will form a pants decom of $X$.

Fact: Since $\operatorname{inj}(X_{\geq \varepsilon}) \geq \varepsilon_0$, we can approximate $X_{\geq \varepsilon}$ by a simplicial complex $\mathcal{S}$, $n$ vertices, $n$ edges, $n$ triangles, each triangle equilateral (Delaunay triangulation of an $\varepsilon_0$-net) and degree bounded.

This is bi-Lipschitz equiv to $X$. 
Number the vertices of $Y$, $y_1, \ldots, y_n$. Let $h : Y \to \mathbb{R}$ be the PL map sending $y_i$ to $i$. We'll do some Morse theory on $h$. If $a = h^{-1}(k+1)$, then $h^{-1}(a)$ and $h^{-1}(b)$ are hyperbolic.

The set $h^{-1}(Z + \frac{1}{2})$ is a union of disjoint closed curves each of length $\leq 9$ (they each pass through each triangle at most once).

This is not a pants decomposition, but it's close! Ex: 

\[ \includegraphics[width=0.25\textwidth]{pants_decomposition.png} \]

It's a pair of pants, (more generally, a little more complicated, but not much.)

Okay, but this is an obviously bad algorithm - it ignores the structure of the surface. But it's asymptotically as good as any algo we have.

Lower bounds:

Busser's hairy torus:

\[ \includegraphics[width=0.25\textwidth]{hairy_torus.png} \]

Glue $N^2$ of these to form $X$:

\[ \includegraphics[width=0.5\textwidth]{gluing.png} \]

Any PD lens a curve going around the torus, so

PL$(X) \sim \sqrt{2N^2} N \approx 9$.

But most of the curves can be taken small – TPL$(X) \sim N^2 + g$.

Arithmetic hyperbolic surfaces: $\exists$ surfaces s.t. $\text{inj}(X) \geq \log(g(X))$ via orbit construction representations of $\text{Isom}(H)$ on $SL_2(\mathbb{R})$.

Then $\text{PL}(X) \geq \log g$, $\text{TPL}(X) \geq g \log g$ (but probably much larger).

Kaplan hyperbolic surfaces: Theorem (Guth-Pardon-Y): $\forall \varepsilon > 0, \exists$ if $X$ is a random genus-$g$ hyp surface and $g$ is sufficiently large then $\text{TPL}(X) \geq 9^{\frac{g}{2} - \varepsilon}$. 
\( PF_i \sim \text{is up to exponential terms in } g \)

Recall \( \text{Vol}(M_g) \sim g^3 \)

The combinatorial type of a PD is the trivalent graph formed by a curve whose vertices are the pants, with one edge for each curve.

If \( G \) is a trivalent graph with \( 2g - 2 \) vertices, let

\[ C^G_{M_g} = \text{set of genus-} g \text{ surfaces with a pants decom} \]

of total length \( \leq L \).

Then \( \text{vol } P(G, L) \) is easy to bound:

\[ \text{vol } P(G, L) \leq \sum_i \int_{L_i + \epsilon}^{L_i - \frac{L_i}{2g-3}} L \]

\[ \leq \left( \frac{L}{g} \right)^6 \text{ (roughly equiv to choosing } \mathcal{C}_f, \mathcal{C}_i \text{ from } [0, \frac{L}{g}] \right) \]

Similarly for the \# of trivalent graphs:

\[ \binom{2g-2}{6g-6} \text{ half-edges} \]

2g-2 vertices \( \binom{2g-2}{6g-9} \) \( \sim 3 \) glueings \( \sim g^{3g} \)

\( (2g-2)! \approx g^{3g} \) labelings

\( \approx g^g \) graphs.

So, \( \text{vol } P(G, L) \leq \frac{L^6}{g^6} g^g \).

If \( L < g^{2g-2} \), this is much less than \( \text{vol } C_{M_g} \approx g^{2g} \).

Open question: Improve way of these bounds.

Another model:

Combinatorial random surface - glue \( N \) triangles at random.

\[ E[\text{# of vertices}] \sim \log N \]

\[ E[\text{genus}] \sim \frac{N}{4} \]

\[ \frac{\text{# of surfaces}}{\text{# of } N \text{ triangles}} \sim \frac{(3N-1)(3N-3)(3N-5)}{N_1} \approx \frac{N}{g^{2g}} \]

Pants decomposes - Combinatorial pants decomposes:

Tight it up

Prop: If a comb. surf has a pants decom w/ total length \( L \), it has a tight PD w/ total length \( \sim L \).
Counting TPDs: There are tight pants with a triangle and specified bdy lengths.

So, $D_{\text{comp}}(G,L) \leq e^{-L}, \quad \text{for } k \geq \left( \frac{3N}{k} \right)^{3/2}.

Here, $K \leq \frac{N}{2}, \text{ so } k \geq \frac{N}{2}.

And the argument is exactly the same as before.

Q: What's the relationship between random hypersurfaces and combinatorial surfaces?

Last time: Random surfaces, via:
- continuous moduli spaces: Teichmüller space
- discrete moduli spaces: random triangle gluings

But the discrete moduli space approximates the continuous one?

Today, 3-manifolds:
Difference: Both these approaches have problems.
- Continuous approach: $\mathbb{R}^3$ has no moduli space of hyperbolic structures. Every hyperbolic manifold has a unique hyperbolic metric.

Theorem (Mostow): If $M, M'$ are hyperbolic manifolds of finite volume with $\pi_1(M) \cong \pi_1(M')$, then $M \cong M'$ are isometric (i.e., hyperbolic equivalence $\Rightarrow$ isometry).

Discrete approach: gluing tetrahedra.

Problem: In 2d, this works because the link of a vertex is a random 1-manifold - a circle. In 3d, the link is a random 2-manifold - typically a high genus surface.

What can we do? It's generally hard, a lot of open questions.
I want to try to summarize.

What do 3-manifolds look like?
Theorem (Perelman-Thurston): Every oriented prime closed manifold can be cut along tori so that the interior of each piece has a complete finite volume geometric structure based on one of eight moduli geometries.