Problem Set 6 (due Apr. 27)

April 20, 2017

1. Show that CAT(0) spaces have the approximate midpoint property. That is, if X is CAT(0) and $x, y \in X$ are points such that $r = d(x, y) > 0$, then for any ϵ, there is a δ such that if $d(z, x) \leq \frac{r}{2} + \delta$ and $d(z, y) \leq \frac{r}{2} + \delta$, then $d(z, m) \leq \epsilon$, where m is the midpoint of x and y.

2. Suppose that X is a Riemannian manifold that is CAT(0). Show that X has nonpositive sectional curvature. It may help to go back to problem set 4.

3. Let $\gamma, \gamma' \colon [0, \infty) \to X$ be unit-speed geodesic rays such that $\gamma(0) = \gamma'(0) = p$. In class, we defined the comparison angle between γ and γ' by

$$\angle_p(\gamma, \gamma') = \lim \sup_{t,t' \to 0} \angle_{\bar{p}}(\overline{\gamma(t)}, \overline{\gamma'(t')}),$$

where $\triangle(\bar{p}, \overline{\gamma(t)}, \overline{\gamma'(t')}$ is the comparison triangle for $\triangle(p, \gamma(t), \gamma'(t))$. That is, by the law of cosines,

$$\cos \angle_p(\gamma, \gamma') = \lim \inf_{t,t' \to 0} \frac{t^2 + t'^2 - d(\gamma(t), \gamma'(t'))^2}{2tt'}.$$

Show that this limit exists (not just the lim sup) when X is CAT(0). (Suppose that $s < t$ and $s' < t'$. How does $\angle_{\bar{p}}(\overline{\gamma(s)}, \overline{\gamma'(s')})$ compare to $\angle_{\bar{p}}(\overline{\gamma(t)}, \overline{\gamma'(t')})$?)