Problem Set 5 (due Apr. 13)

April 12, 2017

1. Suppose that $\gamma\colon [0,1] \to M$ is a geodesic. The second variation formula states that if $V = \frac{d\gamma}{dt}$ and $W_1, W_2 \in \mathcal{V}(\gamma)$ are piecewise-smooth vector fields with $W_i(0) = W_i(1) = 0$, then

$$\frac{1}{2}H(E)(W_1, W_2) = -\sum_t \langle W_2, \delta_t D_t W_1 \rangle - \int_0^1 \langle W_2, D_t^2 W_1 - R(V, W_1)V \rangle \, dt.$$

Show that this can be rewritten in the more symmetric form

$$\frac{1}{2}H(E)(W_1, W_2) = \int_0^1 \langle D_t W_1, D_t W_2 \rangle + \langle R(V, W_1)V, W_2 \rangle \, dt.$$

This expression is known as the *index form.*

2. Let M be a compact n–manifold and suppose that $\mathrm{Ric}(U, U) \geq (n-1)k$ for all $U \in TM$ such that $\|U\| = 1$. Prove that if γ is a geodesic in M of length greater than $\frac{m\pi}{\sqrt{k}}$, then $\operatorname{index}\gamma \geq m$.

3. Recall that the Poincaré disc model of the hyperbolic plane $\mathbb{H}^2$ consists of the open unit disc $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ with the metric $dg^2 = \frac{dx^2+dy^2}{(1-x^2-y^2)^2}$. If we identify $\mathbb{R}^2$ with the complex plane, we can write the metric as

$$dg^2 = \frac{|dz|^2}{(1 - |z|^2)^2}.$$

Prove that for any $a \in D^2$, $\theta \in \mathbb{R}$,

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

is an isometry of $\mathbb{H}^2$.