Problem Set 4 (due Mar. 23)

March 9, 2017

1. Let M be a Riemannian manifold and let d be the distance function on M. Show that M is complete with respect to d if and only if M is geodesically complete.

2. Show that the curvature tensor of a Riemannian metric is determined by its sectional curvatures.

3. (a) Let $p \in M$ and let $V, W \in T_pM$. Let $\gamma = \gamma_V$ be a geodesic such that $\gamma(0) = p$ and $\gamma'(0) = V$. Let J be the Jacobi field on γ such that $J(0) = 0$, $D_tJ(0) = W$. Find $D_t^2 J(0)$ and $D_t^3 J(0)$.

   (It may help to let $E_1, \ldots, E_n \in \mathcal{V}(\gamma)$ be a set of orthonormal parallel fields on γ and write $J = \sum_i f_i E_i$.)

   (b) Let $\epsilon > 0$ be such that $\exp_p \colon B(0, \epsilon) \to U \subset M$ is a diffeomorphism and let $(x^1, \ldots, x^n) \colon U \to \mathbb{R}^n$ be the corresponding normal coordinate system. (See Prop. 5.11 of Lee for some properties of normal coordinates.) Use the calculation above to show that the second-order Taylor series of g with respect to normal coordinates is given by

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} \sum_{kl} \langle R(\partial_i, \partial_k)\partial_l, \partial_j \rangle x^k x^l + O(|x|^3),$$

   where $x = (x^1, \ldots, x^m)$.