Problem Set 3 (due Mar. 2)

February 23, 2017

1. Suppose that $a\colon (M,g) \to (N,h)$ is an isometry.
   - Show that a takes geodesics to geodesics.
   - Suppose that M is connected. Show that if $b\colon (M,g) \to (N,h)$ is an isometry and $x \in M$ is such that $a(x) = b(x)$ and $a_*(v) = b_*(v)$ for all $v \in T_x M$, then $a = b$.

2. Suppose that M and N are connected Riemannian manifolds and that $a\colon M \to N$ is a map that preserves the distance function (i.e., $d_M(x,y) = d_N(a(x),a(y))$ for all $x, y \in M$). Show that a preserves the Riemannian metric (i.e., $\langle v, w\rangle_M = \langle a_*(v), a_*(w)\rangle_N$ for all $x \in M$ and $v, w \in T_x M$). (First show that a is smooth; this can be done using the smoothness of the exponential map.)

3. Suppose that M is a compact Riemannian manifold.
   (a) Prove that there is an $\epsilon > 0$ such that every closed curve of length $< \epsilon$ is null-homotopic.
   (b) Suppose that $\gamma : S^1 \to M$ is a closed curve. Prove that there is a piecewise-smooth curve γ' that is homotopic to γ and such that $\ell(\gamma') \le \ell(\gamma)$.
   (c) If $\gamma : S^1 \to M$ is a closed curve, the free homotopy class of γ is the set of curves that are homotopic to γ. Show that this set has a minimal-length element and that this element is a geodesic.