Problem Set 2

February 23, 2017

Due date: Thursday, February 23

1. In class, we used the fact that the curvature tensor can be expressed in terms of parallel transport. That is, we claimed that if $q \in M$ and if $\alpha\colon \mathbb{R}^2 \to M$ is a smooth map such that $\frac{\partial \alpha}{\partial x} = X$, $\frac{\partial \alpha}{\partial y} = Y$, then for all $Z \in T_qM$, we have
$$R(X,Y)Z = \lim_{s\to 0} \frac{Z - p_{\gamma_s}(Z)}{s^2},$$
where $\gamma_s : [0,4] \to M$ is the image under α of the boundary of an $s \times s$ square, i.e.,
$$\gamma_s(t) = \begin{cases} \alpha(st,0) & t \in [0,1] \\ \alpha(s, s(t-1)) & t \in [1,2] \\ \alpha(s(3-t), s) & t \in [2,3] \\ \alpha(0, s(4-t)) & t \in [3,4]. \end{cases}$$
Prove this fact.

   (Hint: Construct a frame of vector fields $V_1, \dots, V_m \in \mathbf{V}(\alpha)$ such that $\nabla_X V_i = 0$ and $\nabla_Y V_i(u_1, 0) = 0$. Any vector field W along γ_s can be expressed as a linear combination of the V_i — when is W parallel?)

2. Suppose that M is a two-dimensional Riemannian manifold with curvature tensor R.
   (a) Use the symmetries of the curvature tensor to show that if $p \in M$, $V, W \in T_pM$, then
   $$K = \frac{\langle R(V,W)W, V\rangle}{\|V\|^2\|W\|^2 - \langle V, W\rangle^2}$$
   is independent of V and W.
   (b) Prove that if K is as above, then
   $$R(X,Y)Z = K(\langle Y, Z\rangle X - \langle X, Z\rangle Y)$$
   for all $X, Y, Z \in T_pM$.

3. Find a sequence $\gamma_i\colon [0,1] \to S^2$ of piecewise-smooth curves with the same endpoints such that the γ_i converge to a piecewise-smooth curve γ, but the parallel transport maps $P_{\gamma_i}\colon T_pM \to T_qM$ do not converge to the map P_γ.

Let $M \subset \mathbb{R}^3$ be the cone
$$M = \{(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) \mid r > 0, \theta \in [0, 2\pi)\}$$
as in the first problem set.

4. Construct an isometry from the complement of the line $\theta = 0$ to a subset of $\mathbb{R}^2$. Use this isometry to calculate the parallel transport map $P_{0,2\pi}$ from problem 5 of the first problem set.