Problem Set 1

February 6, 2017

Due date: Thursday, February 9

1. Show that the covariant derivative can be expressed in terms of parallel transport in the following sense.

   Let $\gamma : [0,1] \to M$ be a smooth curve in M and let $p_t : T_{\gamma(0)}M \to T_{\gamma(t)}M$ be the parallel transport maps along γ. Show that if $X \in \mathbf{V}(\gamma)$, then

   $$D_t X(0) = \frac{d}{dt} p_t^{-1}(X(t)).$$

2. Prove that the tangential connection ∇^T on an embedded submanifold $M \subset \mathbb{R}^n$ is compatible with the metric induced by the dot product on $\mathbb{R}^n$.

3. Prove that ∇^T is torsion-free. (Since τ is a tensor, it suffices to show that for any patch $(u^1, \ldots, u^n) \colon U \to M$, $\tau(\partial_i, \partial_j) = 0$.)

Let $\phi \in (0, \pi/2)$ and let $M \subset \mathbb{R}^3$ be the cone

$$M = \{(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) \mid r > 0, \theta \in [0, 2\pi)\}$$

with axis the positive z–axis and angle ϕ.

4. The parametrization $u(r, \phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$ has coordinate vector fields $\partial_r = \frac{\partial}{\partial r}$ and $\partial_\theta = \frac{\partial}{\partial\theta}$. Let ∇^T be the tangential connection on M and calculate $\nabla^T_{\partial_r}\partial_r$, $\nabla^T_{\partial_r}\partial_\theta$, $\nabla^T_{\partial_\theta}\partial_r$, and $\nabla^T_{\partial_\theta}\partial_\theta$.

5. Let $\gamma \colon [0, 2\pi] \to M$ be the circle $\gamma(t) = u(1, t)$. Calculate the parallel transport map $P_{0,2\pi} \colon T_{\gamma(0)}M \to T_{\gamma(2\pi)}M$.