(2016-03-23)

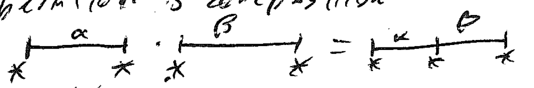
Finish up Morse theory. Last time: $\Omega S^n \simeq X \simeq \{p\} \cup D^{n-1} \cup D^{2(n-1)} \cup \cdots$

Freudenthal Suspension Theorem, Lyusternik-Fet theorem.

Rely on higher htpy gps:

$\pi_1(X) = \{\alpha : (S^1, *) \to (X, *)\} / \text{htpy}$

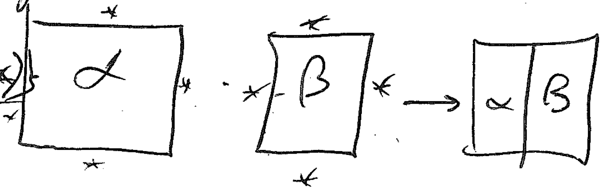
operation is composition:



$\pi_n(X) = \{\alpha : (S^n, *) \to (X, *)\} / \text{htpy}$

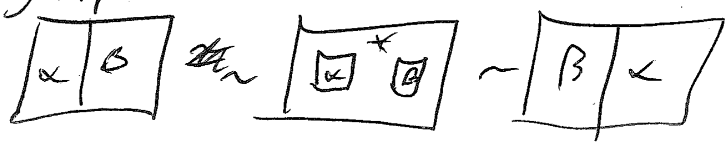
operation:

$= \{\alpha : (~~B^n~~ [0,1]^n, \partial[0,1]^n) \to (X, *)\} / \text{htpy}$



- Measures higher-dim top structure: $\pi_k(S^n) = 0$ if $k < n$, $\pi_k(S^n) = \mathbb{Z}$ if $k = n$

- ~~Simpler in some~~ ~~Simpler group structure~~ π_n is abelian:



- But much harder to compute — we don't know $\pi_k(S^n)$ when $k > n$!

Ex: ~~maps $S^3 \to$~~ $\pi_3(S^2)$? Hopf fibration

$\alpha : S^3 \to S^2$

$S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$

$S^2 = \mathbb{C} \cup \{\infty\}$.

$\alpha(z, w) = \frac{z}{w}$ ~~Then $\alpha^{-1}(x)$ is a circle~~

In fact, ~~then~~ this is a fibration: $\forall x \in S^2$ $\alpha^{-1}(x) \cong S^1$ and $\forall x \in S^2$, $\exists U$ a nbhd of x s.t. $\alpha^{-1}(U) \cong S^1 \times U$.

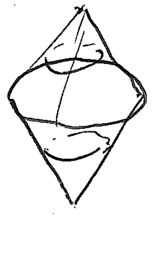
~~Further, if we draw S^3 as $\mathbb{R}^3 \cup \{\infty\}$, then~~

~~Further~~ The Hopf invariant of β is the linking number of

$H(\beta) = \text{link}(\beta^{-1}(x), ~~\text{and}~~ \beta^{-1}(y))$ — this is indep of x, y, htpy inv't, and $H(\text{const}) = 0$, $H(\alpha) = 1$. $\Rightarrow \alpha \not\simeq \text{const}$.

In general, $\pi_k(S^n)$ is not known, but there are patterns — notably:

Suspension:



$\Sigma S^n = S^{n+1}$.

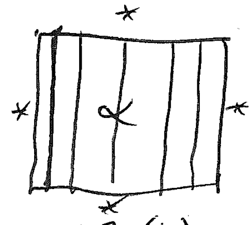
If $\alpha : S^k \to S^n$

$\Sigma\alpha : \Sigma S^k \to \Sigma S^n$ — so $\Sigma : \pi_k(S^n) \to \pi_{k+1}(S^{n+1})$

$: S^{k+1} \to S^{n+1}$

Freudenthal Suspension Theorem: If $k < 2n-1$, then $\pi_k(S^n) \cong \pi_{k+1}(S^{n+1})$

Pf: If ΩX is loop space of X, then
$$\pi_k(\Omega X) \cong \pi_{k+1}(X)$$



So: $\pi_{k+1}(S^{n+1}) \cong \pi_k(\Omega S^{n+1})$

$\cong \pi_k(\{p\} \cup D^n \cup D^{2n} \cup \cdots)$

$\alpha \in \pi_2(X)$

$\alpha_t \in \pi_1(X)$, $t \in [0,1]$

Fact: If X is a CW complex, then the ~~$\pi_k(X) \cong \pi_1$~~ inclusion $X^{(n)} \to X$ induces an isomorphism $\pi_k(X^{(n)}) \to \pi_k(X)$ for all $k < n$. (Analogous to computing π_1 from 2-skeleton). So:

$\pi_{k+1}(S^{n+1}) \cong \pi_k(\{p\} \cup D^n)$ if $k < 2n-1$.

$\cong \pi_k(S^n)$

These are the stable homotopy groups of spheres.

Lyusternik-Fet: If M is a closed Riemannian mfld, it contains a closed geodesic.

Pf: If $\pi_1(M) \neq 0$, then we can find a closed geod as the curve of minimum length in a free htpy class.

Suppose M is simply-connected.
Let ΛM = free loop space $= \{f : S^1 \to M\}$, f piecewise smooth$\}$,
let $E : \Lambda M \to \mathbb{R}$ be energy.
Can we find critical pts of E?

Problem: too many critical pts — every constant path is a critical point.
- really, we want critical pts with positive energy.

Minimax method:

- critical pt is the minimum highest pt of a path from a to b.

Thm: Suppose M is a mfld, $f : M \to \mathbb{R}$ is a smooth fn s.t. $M^a = f^{-1}((-\infty, a])$ is cpt for all a. If $\alpha : S^k \to M$, then
$$y(\alpha) = \inf_{\substack{\beta : S^k \to M \\ \beta \sim \alpha}} \max_{x \in S^k} f(\beta(x))$$
is a critical pt.

Pf: Suppose not. Then $\exists\, \varepsilon > 0$ s.t. f has no critical ~~pts~~ values in $[y - \varepsilon, y + \varepsilon]$. By Morse Lemma, there is a def. retract $r : M^{y+\varepsilon} \to M^{y-\varepsilon}$.
If $\beta \sim \alpha$ and $\max f(\beta(x)) < y + \varepsilon$, then $r \circ \beta \sim \alpha$ and $\max f(r(\beta(x))) < y$ $*$.

Let $M \subset \Omega M$ be the set of const paths. $\exists \varepsilon > 0$ s.t.

Lemma: ~~Any~~ There is a deformation retract $r_\varepsilon : \Omega M^\varepsilon \to M$

Pf: If $E(\alpha) < \varepsilon$, then $L(\alpha) < \sqrt{\varepsilon}$. If ε is suff. small, local existence of geodesics implies that we can homotope α to a point along geodesics; $\alpha(t)$ . This homotopy is continuous on ΩM^ε. ~~and retracts~~

~~Thm~~ Fact: $\pi_k(\Lambda M) \cong \pi_{k+1}(M) \oplus \pi_k(M)$, and there is a minimal k s.t. $\pi_k(M) \neq 0$.

Pf of Lyusternik-Fet: Let k be s.t. $\pi_k(M) \neq 0$, $\pi_{k-1}(M) = 0$.
Let $\alpha \in \pi_k(M)$, $\alpha \neq 0$. ~~Then $\exists$~~ Let $T\alpha \in \pi_{k-1}(\Lambda M)$ be corresp element. Claim: $g(T\alpha) > 0$.
~~Pf:~~ If $g(T\alpha) < \varepsilon$, then ~~$T\alpha$ is homotopic~~ $r_\varepsilon \circ T\alpha \sim T\alpha$, ~~but~~ and $r_\varepsilon \circ T_\alpha \in \pi_{k-1}(M)$. But $\pi_{k-1}(M) = 0 \Rightarrow T\alpha$ constant ✗.

Break. (algorithmic in a way: Find the low-dim nontriv sphere in M. Sweep it out by paths $M(F)$

~~Next: Curvature and~~ Curvature and geometry — how does curvature affect geods, surfaces, symmetries of a space? Study spaces w/ curvature bounds, esp. nonpositively curved spaces.

One of the main theorems we'll use is Toponogov's theorem — if curvature of a space satisfies certain ineqs, then metric satisfies ineqs — and these ineqs are stated in terms of comparison to a ~~model space~~ space of constant curvature.
So.. let's ~~start by looking at spa~~

Spaces of constant curvature

Thm: $\forall \kappa \in \mathbb{R}$, $\forall n \geq 2$ $\exists!$ M^n_κ (up to isometry) s.t. M^n_κ is complete, simply connected, and $\forall x \in M^n_\kappa$, $\forall v, w \in T_x M^n_\kappa$ linearly independent $K(v,w) = \kappa$

- If $\kappa > 0$, $M^n_\kappa = S^n(\frac{1}{\sqrt{\kappa}})$
- If $\kappa = 0$, $M^n_\kappa = \mathbb{R}^n$
- If $\kappa < 0$, M^n_κ = hyperbolic plane.

(Note: Really, we only need three: M^n_1, M^n_0, M^n_{-1}, because scaling by s changes curvature by s^2.

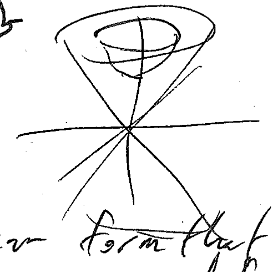
Two are familiar, so I'm just going to list some properties

$\mathbb{R}^n$ — $\mathrm{Isom}(\mathbb{R}^n)$ acts transitively on $\mathbb{R}^n$ w/ pt stabilizer $O(n)$
— $\mathrm{Isom}(\mathbb{R}^n)$ is generated by reflections
— Consequently, lots of totally geodesic subspaces: every plane is the fixed-pt-set of a reflection.

$S^n = \{(x_1, \dots, x_{n+1}) \mid \sum x_i^2 = 1\}$ with metric induced by ~~standard~~ Riemannian metric on $\mathbb{R}^{n+1}$.

— Consequently, isometries of $\mathbb{R}^{n+1}$ descend —
$\mathrm{Isom}(S^n) = O(n+1)$ acts transitively, point stabilizers are $O(n)$
— Generated by reflections
— lots of totally geod subspaces.
— Formula for distance based on norm:
If $x, y \in S^n$, then $\cos d(x,y) = \langle x, y\rangle$.

H^n less familiar - couple different models. But there's one model, hyperboloid model, that is remarkably close to S^n:
note that if $K > 0$,
$$M^n_K = \{(x_1, \dots, x_{n+1}) \mid \sum x_i^2 = \tfrac{1}{K}\}.$$
So, formally not unreasonable to ~~take~~ write
$$M^n_{-1} = \{(x_1, \dots, x_{n+1}) \mid \sum x_i^2 = -1\} -$$
sphere of radius i. And this works if we make one change -
$$M^n_{-1} = \{(x_1, \dots, x_{n+1}) \mid x_1^2 + \dots + x_n^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\}$$



~~= pseudo-Riemannian metric on $\mathbb{R}^{n+1}$~~
~~= Lorentz metric on $\mathbb{R}^{n+1}$~~
~~Let~~ Quadratic form coming from bilinear ~~form~~ that is not pos. def.

Metric on H^n? Same as sphere: Let $\langle v, w\rangle_L = v^T \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} w$.

$\forall x \in H^n$
Lemma: $T_x H^n = x^\perp$
Pf: If $v \in \mathbb{R}^{n+1}$ then $\frac{d}{dt} \|x + tv\|_L^2 = \frac{d}{dt} \langle x+tv, x+tv\rangle_L = 2\langle x+tv, v\rangle_L$

Then $v \in T_x H^n \iff 2\langle x+tv, v\rangle_L \big|_{t=0} = 0$
$\Rightarrow \langle x, v\rangle_L = 0$..

Lemma: $\langle\ ,\ \rangle_L$ is pos. def on $T_x H^n$ $\forall x \in H^n$.
Pf: Index $(\langle\ ,\ \rangle_L) = 1$. and $\langle x, x\rangle_L < 0$, so pos def on ~~orthog~~ $x^\perp$.

Cor: $\langle\ ,\ \rangle_L$ induces a Riemannian metric on H^n.
~~Lem: $\mathrm{Isom}(H^n) = \{M \in GL_{n+1} \mid M^T \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} M = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}\} = O(n,1)$~~
~~M pres~~ sends H^n to H^n

Cor: If $M \in GL_{n+1}$, $MH^n = H^n$, and $\langle v,w\rangle_L = \langle Mv, Mw\rangle_L$ $\forall v,w \in H^n$ then M is an isometry of H^n.

2016-03-30

Away from differential, more toward geometry — How does curvature affect the usual objects of geometry — curves, surfaces, triangles, symmetries, etc.?
~~There are lots of directions to go here — but the direction that I'm most interested in is not~~ ~~Especially nonpositively curved spaces.~~

Simplest spaces where we can see effects of curvature: ~~model spaces~~ Spaces of constant curvature (model spaces)

Thm: $\forall \kappa \in \mathbb{R}$, $\forall n \geq 2$, $\exists! M^n_\kappa$ (up to isometry) s.t. M^n_κ is simply-connected, complete, and $K(v,w) = \kappa$ $\forall$ linearly independent $v,w \in T_x M^n_\kappa$

- If $\kappa > 0$, $M^n_\kappa \cong S^n\left(\frac{1}{\sqrt{\kappa}}\right)$
- If $\kappa = 0$, $M^n_\kappa \cong \mathbb{R}^n$
- If $\kappa < 0$, $M^n_\kappa \cong$ hyperbolic plane.

(In fact, scaling by s multiplies curvature by s^2, so all M^n_κ are scalings of M^n_1, M^n_0, M^n_{-1})

Uniqueness is not hard to prove — use Jacobi fields to ~~construct~~ calculate the metric in polar coords. Easiest way to prove Existence is to construct.

$\mathbb{R}^n, S^n$ are familiar, so let me just list some properties:

$\mathbb{R}^n$ — $\mathrm{Isom}(\mathbb{R}^n)$ acts transitively on $\mathbb{R}^n$ with pt stabilizer $O(n)$
- $\mathrm{Isom}(\mathbb{R}^n)$ generated by reflections
- Consequently, lots of totally-geodesic subspaces — every plane is the fixed-pt set of a reflection.

$S^n = \{(x_1, \dots x_{n+1}) \mid \sum x_i^2 = 1\}$ (all others are scalings)
(get a lot of properties from $\mathbb{R}^{n+1}$):
- $\mathrm{Isom}(S^n) = O(n+1)$ acts transitively with $O(n)$ pt. stabilizers
- $\mathrm{Isom}(S^n)$ generated by reflections
- S^n has many totally geodesic subspaces.
- Can write distance fn on S^n in terms of norm on $\mathbb{R}^{n+1}$ — $\forall x,y \in S^n$, $d(x,y) = \cos^{-1}\langle x,y\rangle$.

H^n is more complicated — several different models, depending on what you want to highlight. But the most elegant one is that we can define it in a way that's awfully close to S^n —

If $\kappa > 0$, $M^n_\kappa = \{(x_1, \dots x_{n+1}) \mid \sum x_i^2 = \frac{1}{\kappa}\}$. So, formally, we could imagine
$$M^n_{-1} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = -1\}$$
— the sphere of radius i. And, if we make one change, this works — instead of the ~~inner~~ Euclidean norm here, we use Lorentz norm.

$H^n = M^n_{-1} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_n^2 - x_{n+1}^2 = -1, \ x_{n+1} > 0\}$

This is a hyperboloid w/ two sheets so we pick the positive one.

Let $\mathbb{R}^{n,1} = \mathbb{R}^{n+1}$, equipped with form $\langle v, w \rangle_L = v^t \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} w$.

Then $H^n = \{v \in \mathbb{R}^{n,1} \mid \|v\|_L^2 = \langle v, v \rangle_L = -1\}$

- this is a form with signature $(n,1)$ - i.e., index 1, nullity 0

(in fact any other such form is the same up to change of basis)

Prop: $\langle , \rangle_L$ induces a Riemannian metric on H^n.

Pf: ~~(Claim: positive definite~~

WTS: $\langle , \rangle_L$ is pos. def. on each tangent space.

~~If $x \in H^n$, then~~

Lem: If $x \in H^n$, $T_x H^n = x^\perp$.

Pf: If $v \in \mathbb{R}^{n,1}$, consider $\frac{d}{dt} \|x + vt\|_L^2 = \frac{d}{dt} \langle x+vt, x+vt \rangle_L = 2 \langle x + vt, v \rangle_L$

$\frac{d}{dt} \|x + vt\|_L^2 \Big|_{t=0} = 2 \langle v, x \rangle_L$

$\Rightarrow v \in T_x H^n \iff \langle v, x \rangle_L = 0$ //.

Then $\langle , \rangle_L$ is neg. def on x and index 1, so it is pos. def on $x^\perp$.

~~Cor: Isom$(H^n) \supset$~~ Let $O(n,1) = \{M \in GL(n+1) \mid M \text{ preserves } \langle , \rangle_L\}$

~~$= \{M \in GL(n+1) \mid \langle Mv, Mw \rangle_L = \langle v, w \rangle_L\}$~~

$= \{M \in GL(n+1) \mid v^t \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} w = v^t M^t \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} M w\}$

$O(n,1)_+ = \{M \in O(n,1) \mid MH^n = H^n\}$ - M doesn't swap sheets of the hyperboloid. - Clearly $O(n) \subset O(n,1)_+$.

Further, $\begin{pmatrix} a & \sqrt{a^2-1} \\ \sqrt{a^2-1} & a \end{pmatrix} \in O(1,1)_+$, ~~then $O(n,1)_+ \subset$ Isom(H^n)~~ so $O(1,1)_+$ acts transitively on H^1

$\Rightarrow O(n,1)_+$ acts transitively on H^n with $O(n)$ point stabilizers.

Prop: Isom$(H^n) = O(n,1)_+$

Pf: Any isom is determined by image of basepoint, derivative of basepoint. Since $O(n,1)_+$ is transitive on frames, ~~every~~ it contains every isometry.

~~Cor: H^n has constant sectional curvature~~ $\boxed{\text{break}}$

Pf: ~~K is invariant under isometries, and Isom acts transitively on 2-planes.~~ So, H^n is highly symmetric, ~~def~~ level sets of a quadratic form - what else?

~~Distance formula:~~

Distance formula ~~is~~: Recall if $x, y \in S^n$, $d(x,y) = \cos^{-1} \langle x, y \rangle$.

Nearly the same here: rotate so ~~x~~ $\mathbb{R}^{2,1}$

If $x, y \in H^n$, ~~WLOG~~, $x = (0, \dots, 0, 1)$ ~~$\langle x, y \rangle_L = y_2$~~

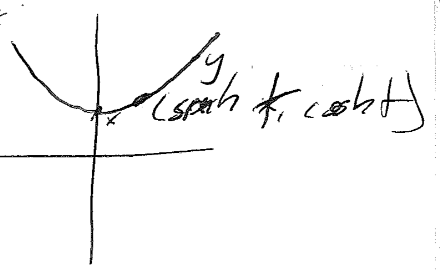
$y = (0, \dots, y_1, y_2)$

~~and~~ we can look at $\int^{\cosh^{-1} y_2}$ ~~So take~~

Then $d(x,y) = \int_0^{\cosh^{-1} y_2} \sqrt{\|(\cosh^2 t, \sinh^2 t)\|_L}\, dt$

$= \int_0^{\cosh^{-1} y_2} dt = \cosh^{-1} y^2$

$= \cosh^{-1} -\langle x,y\rangle_L$.



Geodesics are intersections - one between each pair of boundary pts

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Thm: ~~If $x \in H^n$, $v,w \in T_x H^n$, then $K(v,w) = -1$~~

~~Pf: Enough to~~ ~~If $x \in H^n$, $v, w \in T_x H^n$, then $K(v,w) = -1$~~

Pf: ~~Using Jac~~

~~Recall: We used Jacobi fields to put polar coords on~~

~~We can identify the model spaces by the circumference of circles:~~

~~We used Jacobi fields to show that~~

~~in $\mathbb{R}^n$ $L(S_1(r)) = 2\pi r$~~

~~in M^n_K $L(S$~~

~~Thm: For $\forall\ K>0$, $\forall$ $n \geq 2$, $\exists!$ complete, simply-connected M^n_K with constant sectional curvature K, and $M^n_0 = \mathbb{R}^n$, $M^n_1 = S^n$, $M^n_{-1} = H^n$.~~

~~Pf: Jacobi eq calculates the metric:~~

~~$D_t^2 W = R(V,W)V$~~

~~Let such γ is a~~

Thm: ~~H^n has con~~ $H^n = M^n_{-1}$

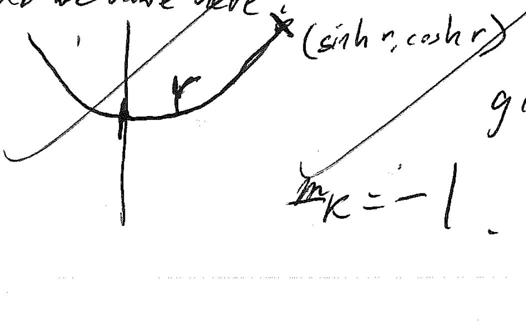
Pf: ~~Clearly~~ $\operatorname{Isom}(H^n)$ acts transitively on 2-planes, so ~~K is~~ H^n has const. sectional curv.

~~Calculate:~~ We can identify model spaces by the circumference of circles:

If $X = M^n_K$, if $V \subset T_x X$ is a 2plane, $\gamma: S^1 \to V$ is circle of radius ~~unit circle~~ in V, we can use Jacobi fields to calculate $\|\exp_* \frac{d\gamma}{dt}\|$ —

$$L(S_1(r)) = 2\pi r \quad \text{if } K = 0$$
$$L(S_1(r)) = 2\pi \frac{\sin(r\sqrt{K})}{\sqrt{K}} \quad \text{if } K > 0$$
$$L(S_1(r)) = 2\pi \frac{\sinh(r\sqrt{-K})}{\sqrt{-K}} \quad \text{if } K < 0.$$

Which one do we have here?



~~Rotat~~ rotating around z-axis gives circle of length $2\pi \frac{\sinh(r\sqrt{1})}{\sqrt{1}}$.

$\Rightarrow K = -1$.

Thm: $M^n_{-1} = H^n$

Pf: $\mathrm{Isom}(H^n)$ acts transitively on the 2-planes, so it suffices to show $K(V,W) = -1$ for some V, W. — lets say at $x_0 = (0,\dots,0,1)$.

~~Let E_1, E_2 be~~

We can measure sectional curvature ~~from~~ the circumference of small circles:

~~if $E_1, \dots, E_n \in T_{x_0}H^n$ is a frame,~~ ~~let consider~~ ~~let $E_1, \dots, E_n \in T_{x_0}H^n$ be a frame~~

~~let $\gamma(t) = \exp_{x_0}(rE_1 \cos t + rE_2 \sin t)$~~ if V, W orthonormal

~~then~~ let $\gamma_r(t) = \exp_{x_0}(r\,\cancel{E_1}V \cos t + r\,\cancel{E_2}W \sin t)$, $t \in [0, 2\pi]$ then

$$\frac{d^3}{dr^3} L(\gamma_r)\Big|_{r=0} = -2\pi K(\cancel{E_1}V, \cancel{E_2}W)$$

(essentially the same calc we did about how things look from far away)

In this case, by the distance formula, γ_r is a circle of (Euclidean) radius $\sinh r$, in the plane $x_{n+1} = \cosh r$.

Then $L(\gamma_r) = \cancel{\sinh}\ 2\pi \sinh r$ (i.e., exponential growth)

and $\frac{d^3}{dr^3} L(\gamma_r)\Big|_{r=0} = 2\pi \cosh 0 = 2\pi$. $\Rightarrow K = -1$. //

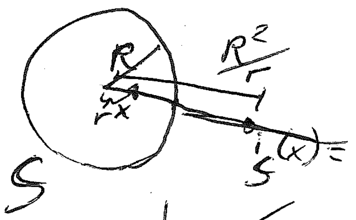
~~Geometry~~: Inversive models: ~~This~~ The hyperboloid model is good for calculations, but it's hard to get a feeling for what the space is like. So let's look at some models that preserve more of the geometry. Two in particular: disc model, upper half-space model.

~~- conformal models - inversive models - isometries are generated by a map inversion~~ ~~isometries generated by inversions~~.

- Disc model: $H^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$

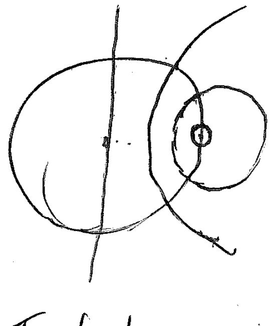
  metric $ds^2 = \frac{4}{(1-r^2)^2} dx^2$.

- conformal model: angles ~~in~~ are preserved by the metric.
- inversive model: ~~geodes~~ isometries are generated by inversions:



Inversions are

- conformal
- order 2
- send circles and lines to circles and lines
- If S is orthogonal to ∂H^n, then inversion through S is an isometry;
- These ~~isometries~~ inversions generate $\mathrm{Isom}(H^n)$ ~~(these are the~~

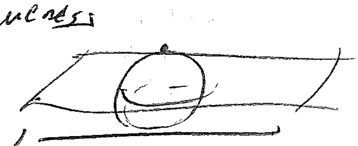


- clearly, they generate ~~fixed~~ ~~st~~ stabilizer of origin
- ~~one~~ point - transitivity on H^n by composing two.

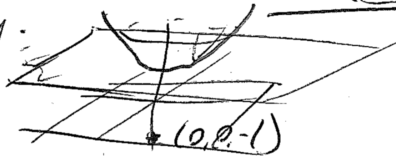
In fact, given two of these that don't intersect, there's a ~~ha~~ geodesic ~~t~~ orthogonal to both — composing the corresponding inversions is like a translation (hyperbolic isometry)

- Geodesics are arcs of circles

- And this is the same as the hyperboloid model
  - 1 way: calculate curvature, & use uniqueness
  - Alt: ~~Ster~~ Stereographic projection



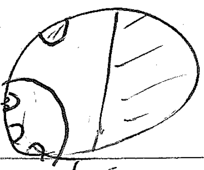
  - Same thing works for hyperboloid:



is an isometry

(~~out of time~~: Important thing: H^n is big. In $\mathbb{R}^n$, half-spaces are big — in H^n, not so much:

-exponential growth



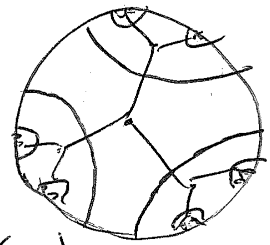
~~We'll see more later but (H^n is very treelike).~~

~~Other model:~~ ~~even simpler metric~~

~~$H^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$~~

~~$dx^2 = \frac{1}{x_n^2} ds^2$.~~

- H^n is tree-like



Ex: Construct complete binary tree in H^2.
  - This is an embedding of tree in H^2 $i: T \to H^2$
  - this is biLipschitz: $c^{-1} d_T(x,y) \le d_{H^2}(i(x), i(y)) \le c\, d_T(x,y)$
  (we'll see later: any two geods have a common perpendicular)
  - Not possible in $\mathbb{R}^n$

Ex: Thin triangles. - in tree, in H^2.

2016-04-06

Last time: Hyperbolic space is tree-like:

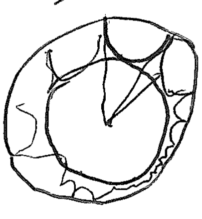
Ex: 

Lots of half-spaces

Ex: Boundary at infinity of a tree is a Cantor set

Any two pts connected by a geod.

Same for hyp.

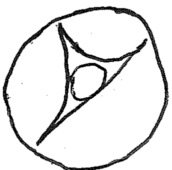


Ex: Triangles are thin  r = inradius(Δ)

And the maximum isn't achieved:

– as the inradius approaches $\log\sqrt{3}$, the vertices go to infinity –

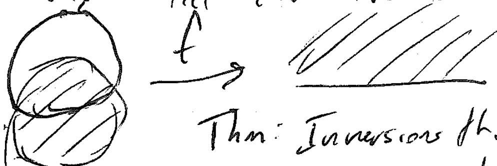
The triangle converges to an ideal triangle, with three vertices on $\partial_\infty H$



To study these, let's introduce another model: upper half space

$H^n = \{(x_1, \dots, x_n) \mid x_n > 0\}$ $\quad ds^2 = \dfrac{dx_1^2 + \dots + dx_n^2}{x_n^2}$

This has a lot of the same structure as disc model — still conformal, still inversive (in fact, image of an inversion:)



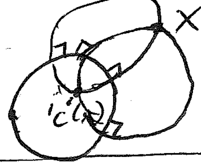
~~So inversions that preserve the boundary are isometries~~

Thm: Inversions that fix the bounding plane are isometries.

Pf: If C is a sphere ~~that~~ centered on the $x_n = 0$ plane, then $f(C)$ is a sphere orthogonal to ∂D^n

Then $f \circ i_{f(C)} \circ f^{-1}$ is an isometry — claim: $f \circ i_{f(C)} \circ f^{-1} = i_C$

But i_C fixes C pointwise, i_C is conformal, sends circles and lines to circles and lines: — this determines i_C completely.



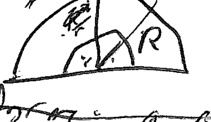
(alternatively, it's easy to calculate)

Pf: Calculate for unit circle. The rest are conjugates (scalings)

~~Alternatively, do a calculation:~~

~~Advantages Properties: Advantages~~ Properties:

- different "obvious" set of symmetries — translations, reflections, rotations preserving ~~parallel to~~ boundary
- scaling — composition of inversions is a scaling



~~These form a group, and that group fixes the point at ∞, i.e.~~ a point on the boundary.

- Geodesics are semicircles, vertical lines

~~Easier to do~~

— ~~study the boundary at infinity~~ : ~~we can see very clearly that~~

- ~~Every~~ Isoms of H^n extend to maps on $\mathbb{R}^n \cup \{\infty\}$ —
  in fact, $\mathrm{Isom}(H^n) \cong$ Möbius transformations of $\mathbb{R}^n \cup \{\infty\}$
  (group generated by all inversions)

— If $n=2$, orientation-preserving isometries are fractional linear transformations
$z \mapsto \frac{az+b}{cz+d}$ where $a,b,c,d \in \mathbb{R}$, $ad-bc=1$.

~~ideal triangles~~ are all congruent. — ~~the~~ i.e. translations $\frac{z+b}{1}$, scalings $\frac{az}{a^{-1}}$, etc.

and $\mathrm{Isom}(H^2) \cong SL_2(\mathbb{R})$. (~~And we use this boundary action to~~ classify:)

~~Thm: Isometries of H^n can be classified into three types.~~
- ~~elliptic: — conjugate to a rotation/reflection of the unit disc.~~
- ~~parabolic: — conjugate to a translation of the half-space model.~~
- ~~hyperbolic — conjugate to a scaling + rotation/reflection of half-space.~~

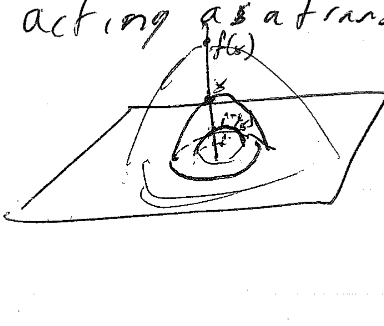
~~Pf~~: We can use the boundary action to study the space.

— ~~Ex~~ All ideal triangles are congruent (the boundary action is 3-transitive)

— We can classify isometries by their fixed points:

Thm: Isometries of H^n fall into three classes:
- elliptic — conjugate to a rotation/reflection of unit disc
- parabolic — conjugate to a translation of half-space model
- hyperbolic — conjugate to a scaling + rotation/reflection of half-space.

Pf: If $f \in \mathrm{Isom}(H^n)$, then ~~f acts on D^n~~ f extends to a homeo $f: \bar{D}^n \to \bar{D}^n$. By Brouwer, it has at least one fixed point.
- If $f(x)=x$ for some $x \in D^n$, f is elliptic
- If ~~$f(x)$~~ f has no fixed pts in D^n, exactly one on ∂D^n, conjugate that to ∞ in half-space model. ~~It's an translation affine map~~
  — Then f sends vertical lines to vertical lines — in fact, planes to planes ~~$(\vec{x}, y) \Rightarrow$ f is a scaling, rotation, translation~~
  $\Rightarrow$ f acts on $\mathbb{R}^{n-1}$ as a scaling/rotation/translation.
  $f(\vec{x}) = A\vec{x}+\vec{v}$ $\forall x \in \mathbb{R}^{n-1}$
  No fixed pts $\Rightarrow$ $A = I_{n-1}$, so f is a translation
- If f has two fixed pts on ∂D^n, x and y, conjugate to $x=\vec{0}$, $y=\infty$.
  Then f acts on $\mathbb{R}^{n-1}$ as an affine map preserving 0 — linear map
  ~~f preserves~~ Let γ be geod from x to y, then f preserves γ, acting as a translation (by log of the scaling factor.)



- If f has >2 fixed points on ∂D^n, it fixes an ideal triangle Δ $\Rightarrow$ f fixes the circumcenter of Δ $\Rightarrow$ f is elliptic. //