2016-04-13

Q: How do props of eucl/hyp space carry over to $K \le 0$ / $K < 0$?

Today: Rauch Comparison thm, geometric implications.

Recall 2VF: If γ is a geod, $W_1, W_2 \in \mathcal{V}(\gamma)$, $W_i(0) = W_i(1) = 0$, then
$$\tfrac{1}{2} H(E)(W_1, W_2) = -\Sigma \langle W_2, \Delta_t D_t W_1 \rangle - \int_a^b \langle W_2, D_t^2 W_1 - R(V, W_1)V \rangle dt.$$

Problem: Not ~~well-defd~~ symmetric, Hessian not well-def'd if W_i do not vanish at endpts. So we define

index form: $I(W_1, W_2) = \int_a^b \langle D_t W_1, D_t W_2 \rangle + \langle R(V, W_1)V, W_2 \rangle dt$.

Prop: I is well-defined, symmetric, bilinear, and if $W_i(0) = W_i(1) = 0$, then $I(W_1, W_2) = \frac{1}{2} H(E)(W_1, W_2)$.

Pf: ETS: $\int_a^b \langle W_2, D_t^2 W_1 \rangle = -\Sigma \langle W_2, \Delta_t D_t W_1 \rangle - \int_a^b \langle D_t W_1, D_t W_2 \rangle dt$.

If W_1, W_2 smooth on $[c, d]$ then
$$\int_c^d \langle W_2, D_t^2 W_1 \rangle = \langle W_2, D_t W_1 \rangle \Big|_c^d - \int_c^d \langle D_t W_1, D_t W_2 \rangle dt$$
$$= \langle W_2(d), D_t W_1(d^-) \rangle - \langle W_2(c), D_t W_1(c^+) \rangle - \int_c^d \langle D_t W_1, D_t W_2 \rangle dt.$$

If W_1, W_2 piecewise smooth on $a = a_1 < \cdots < a_n = b$, then
$$\int_a^b \langle W_2, W_1'' \rangle = \sum_{i=2}^{n} \langle W_2(a_i), W_1'(a_i^-) \rangle - \sum_{i=1}^{n-1} \langle W_2(a_i), W_1'(a_i^+) \rangle - \int_c^d \langle W_1', W_2' \rangle dt$$
$$= \langle W_2(b), W_1'(b^-) \rangle - \langle W_2(a), W_1'(a^+) \rangle - \Sigma \langle W_2, \Delta_t W_1' \rangle - \int_c^d \langle W_1', W_2' \rangle dt.$$

If W_i vanish at endpts these boundary terms vanish. //

Further, this satisfies:

First Index Lemma: ~~If~~ $\gamma : [0, 1] \to M$ ~~be~~ a ~~unit speed~~ geodesic with no conj. pts and ~~let~~ $W \in \mathcal{V}(\gamma)$ is a v. field with $W(0) = 0$, ~~then~~ Let J be the Jacobi field s.t. $J(0) = 0$, $J(1) = W(1)$. Then
$$I(J, J) \le I(W, W), \text{ with equality iff } J = W.$$

Pf (Note: Usually this is a lemma used to prove Morse Index — but since we already proved Morse Index, it'll just follow as a consequence.)

~~Let~~ Extend γ ~~to~~ to a geodesic $\bar\gamma : [0, 1+\varepsilon] \to M$, and extend $\bar W, \bar J$ to $\bar\gamma$ ~~by a Jacobi field s.t.~~ $\gamma|_{[1,1+\varepsilon]}$, so that $\bar J|_{[1,1+\varepsilon]} = \bar W|_{[1,1+\varepsilon]}$.

Then: $I(\bar J, \bar J) = I(J, J) + \underbrace{I(\bar J|_{[1,1+\varepsilon]}, \bar J|_{[1,1+\varepsilon]})}_{C}$

$I(\bar W, \bar W) = I(W, W) + C$.

ETS: $H(E)(\bar W, \bar W) \ge H(E)(\bar J, \bar J)$. Let $\bar W_0 = \bar W - \bar J$.

Then $H(E)(\bar J, \bar W_0) = 0$ (because $\bar W_0 = 0$ outside $[0, 1]$ and $\bar J$ is Jacobi on $[0, 1]$)

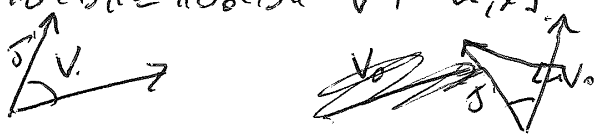
so $H(E)(\bar W, \bar W) = H(E)(\bar J, \bar J) + H(E)(\bar W_0, \bar W_0)$

By Morse Index, $H(E)$ is positive definite $\Rightarrow H(E)(\bar W, \bar W) \ge H(E)(\bar J, \bar J)$.

Rauch Comparison Theorem:

Let M, M_0 be Riemannian manifolds with $\dim M_0 \geq \dim M$, $\gamma, \gamma_0 : [0,l] \to M, M_0$ unit-speed geodesics with $V=\gamma'$, $V_0=\gamma_0'$.
Suppose that $\forall t \in [0,l]$, $\forall X \in T_{\gamma(t)}M$, $X_0 \in T_{\gamma_0(t)}M_0$, $K(X,V) \leq K(X_0,V_0)$. Suppose no conjugate pts to $\gamma_0(0)$ along γ_0.

Then: If J, J_0 are ~~normal~~ Jacobi fields on γ, γ_0 with $J_0(0)=J(0)=0$, $\langle V(0), J'(0)\rangle = \langle V_0(0), J_0'(0)\rangle$, $\|J'(0)\| = \|J_0'(0)\|$, then $\|J(t)\| \geq \|J_0(t)\|$ $\forall t \in [0,l]$. (i.e., magnitude of J', angle with V are equal)



Pf: First: Recall that $J(t) = J^{\perp}(t) + \langle V(0), J'(0)\rangle \cdot t \cdot V(t)$ where $J^{\perp}$ is a normal Jacobi field. So
$$\|J(t)\|^2 = \|J^{\perp}(t)\|^2 + t^2\langle V(0), J'(0)\rangle^2$$
and its enough to consider the case that $J = J^{\perp}$, $J_0 = J_0^{\perp}$.

So: let $f(t) = \frac{\|J(t)\|^2}{\|J_0(t)\|^2}$ — claim that $f'(t) \geq 0$, $\lim_{t\to 0} f(t) = 1$.

First is easy: by L'H, $\lim_{t \to 0} f(t) = \lim_{t\to 0} \frac{\langle J'(t), J'(t)\rangle}{\langle J_0'(t), J_0'(t)\rangle} = 1$.

2: $f'(t) \geq 0 \iff \frac{\langle J'(t), J(t)\rangle}{\langle J(t), J(t)\rangle} \geq \frac{\langle J_0'(t), J_0(t)\rangle}{\langle J_0(t), J_0(t)\rangle}$.

Let $\tau \in [0,l]$. Normalize by letting $W(t) = \frac{J(t)}{\|J(\tau)\|}$, $W_0(t) = \frac{J_0(t)}{\|J_0(\tau)\|}$
ETS: $\langle W'(\tau), W(\tau)\rangle \geq \langle W_0'(\tau), W_0(\tau)\rangle$

Let $E_1, \dots, E_n$ be ~~v. field~~ ortho frame fields on γ, $E_1 = V$, $E_2(\tau) = W(\tau)$
$E_1^0 \dots E_n^0$ " " γ_0. $E_1^0 = V_0$, $E_2^0(\tau) = W_0(\tau)$.

We can use these to translate W to γ_0:
let $W(t) = \sum w_i(t) E_i$, $\widehat{W}_0(t) = \sum w_i(t) E_i^0$
(We lose structure - probably not Jacobi, but that's exactly what we want)

- $\langle \widehat{W}_0, V_0 \rangle = 0$ - $\|W\|^2 = \|\widehat{W}_0\|^2 = \sum w_i^2$
- $\widehat{W}_0(\tau) = W_0(\tau)$ - $\langle \widehat{W}_0, \widehat{W}_0' \rangle = \langle W, W' \rangle$

So: $\langle W_0(\tau), W_0'(\tau)\rangle = \int_0^\tau \langle W_0', W_0'\rangle + \langle W_0, W_0''\rangle \, dt$
$= \int_0^\tau \langle W_0', W_0'\rangle + \langle R(V, W_0)V, W_0\rangle \, dt$ (Jacobi)
$= \cancel{\int_0^\tau \langle W_0', W_0'\rangle} \; I(W_0, W_0)$.

OTOH: $\langle W(\tau), W'(\tau)\rangle$ ~~$= \int_0^\tau \langle W_0', W'\rangle + \langle W_0, W_0''\rangle$~~

$= \int_0^\tau \langle W', W'\rangle + \cancel{\langle R(V,W)V, W\rangle dt}\ K(V,W)\|W\|^2 dt$

$\geq \int_0^\tau \langle \hat{W}_0', \hat{W}_0'\rangle + K(V, \hat{W}_0)\|\hat{W}_0\|^2 dt$

$= I(\hat{W}_0, \hat{W}_0) \geq I(W_0, W_0)$ //

(break). (Did you do the exercise?)

~~Corollary: If γ_0~~

Note: we only assume that γ_0 has no conjugate pts. So —

Cor: If $K(X,Y) \leq \kappa$ $\forall X, Y \in TM$ ($K_M \leq \kappa$), then $\forall x \in M$, $\exp_x|_{B(\frac{\pi}{\sqrt{\kappa}})}$ is nonsingular

(Compare with Myers' Thm from before, which said that if $Ric_M < \kappa(m-1)$, then every geod of length $< \frac{\pi}{\sqrt{\kappa}}$ has conjugate pts)

Cor: ~~Let~~ M, M_0 be s.t. $K_M \leq K_{M_0}$, ~~$x \in M, x_0 \in M_0$, and let $i: T_{x_0}M_0 \to T_xM$ be a norm-preserving map. Then Suppose that $r>0$~~ is s.t. ~~$\exp_x|_B$~~. Suppose $m_0 \in M_0$, $r>0$ s.t. $\exp_{m_0}|_{B_r(0)}$ is a diffeo ~~(in particular, $r < \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$)~~. Let $m \in M$, and let $i: T_{m_0}M_0 \to T_mM$ be an isometry. ~~Then~~ Let $f: B_r(m_0) \to B_r(m)$

$f(x) = \exp_m \circ i \circ \exp_{m_0}^{-1}(x)$. Then for all paths λ in $B_r(m_0)$, $L(f(\lambda)) \geq L(\lambda)$.

Pf: Let $\tilde{\lambda}(t) = \exp_{m_0}^{-1}(\lambda(t))$, let $\alpha_0(u,v) = \exp_{m_0}(v\tilde{\lambda}(u))$, $\in M_0$

Then $J_0 = \frac{\partial \alpha_0}{\partial u}\Big|_{u=\tau}$ is a Jacobi field on $\gamma_0(t) = \exp_{m_0}(t\tilde{\lambda}(\tau))$ with $J_0(0) = 0$, $J_0(1) = \lambda'(\tau)$, $J_0'(0) = \tilde{\lambda}'(\tau)$.

Likewise, if $\alpha(u,v) = \exp_m(i(v\tilde{\lambda}(u)))$, then ~~$J = \frac{\partial\alpha}{\partial u}\big|$~~ $\alpha(t,1) = f(\lambda(t))$ and $J = \frac{\partial \alpha}{\partial u}\Big|_{v=\tau}$ is Jacobi with

$J(0) = 0$, $J(1) = (f\circ\lambda)'(\tau)$, $J'(0) =$ ~~$(f\circ\tilde{\lambda})'(\tau)$~~ $= i(\tilde{\lambda}'(\tau))$.

— so Rauch applies and $\|J(1)\| \geq \|J_0(1)\|$

$\|(f\circ\lambda)'(\tau)\| \geq \|\lambda'(\tau)\|$

$L(f\circ\lambda) \geq L(\lambda)$.

So, if $K_M \leq 0$, then $\exp$ is length-increasing ~~there is a length increasing map from $\mathbb{R}^m \to M$~~ around end

~~This is cumbersome to use, but there's a beautiful condition that $\forall g \in R$, Use this to prove the CAT(0)-condition~~

Seems a bit complicated to use, but we'll use this to prove that that $K_M \leq 0$ implies the if $K_M \leq 0$, then M satisfies CAT(0) condition.

Def: ~~If~~ A *geodesic triangle* is a loop composed of three geodesic segments - we'll write ~~Δ~~ $[p,q]$ for the segments, ~~$[p,q]$~~ $\Delta(p,q,r)$ for the triangle.

If Δ is a geodesic triangle, the comparison triangle $\overline{\Delta} = \Delta(\overline{p}, \overline{q}, \overline{r})$ is ~~the~~ triangle in $\mathbb{R}^2$ with side lengths ~~$L[p,q]$~~ $d(p,q)$, $d(q,r)$, $d(p,r)$

~~Ex:~~ If $x \in [p,q]$, ~~let~~ the comparison point $\overline{x} \in [\overline{p}, \overline{q}]$ is the pt st. $d(p,x) = d(\overline{p},\overline{x})$

~~Then M~~ If X is a geodesic metric space, we say X is CAT(0) if $\forall$ geod tris. Δ, $\forall x,y \in \Delta$, ~~d~~

$$d(x,y) \le d(\overline{x}, \overline{y}).$$

— every measurement of ~~the~~ triangle in X is bounded by the corresp measurement in $\mathbb{R}^2$

Note: No ~~cond~~ requirement that X is a manifold!
In fact, trees are CAT(0) — check this.

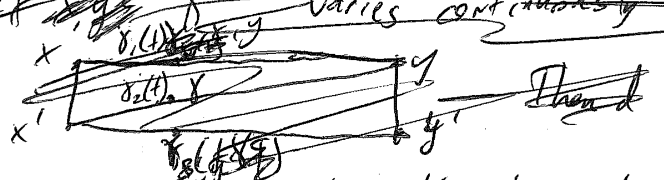
Properties: $\forall x,y \in X$,

— There is a unique geodesic $\gamma_{x,y}$ from x to y

Pf: ~~Suppose not:~~
Suppose ~~$[p,q]$~~ γ, γ' geods from p to q.

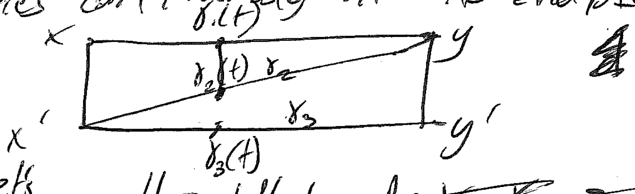
~~is a ges~~ is a geod triangle, so . . .

— ~~(Alt) If x, y~~ ~~γ varies continuously with its endpts:~~

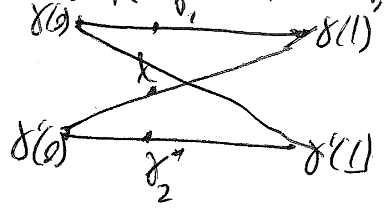


~~Then d~~

— γ_{xy} varies continuously with its endpts: ~~In fact,~~



In fact, let's prove ~~that~~ that ~~$d: X \times X \to \mathbb{R}$~~ the distance function is convex: let γ_1, γ_2 be geodesics, let $t \in [0,1]$
~~let γ_3 be the geod from $\gamma_2(0)$ to $\gamma_1(1)$~~



Claim: $\forall t \in [0,1]$
$$d(\gamma_1(t), \gamma_2(t)) \le (1-t)\, d(\gamma_1(0), \gamma_2(0)) + t\, d(\gamma_1(1), \gamma_2(1)).$$

Pf: let λ be the geod from $\gamma_1(0)$ to $\gamma_2(1)$.
Then $\Delta(\gamma(0), \gamma(1), \gamma'(0))$ is a comparison triangle.
by Similar tris, $d(\gamma_1(t), \lambda(t)) \le (1-t)\, d(\gamma_1(0), \gamma_2(0))$
Likewise, $d(\lambda(t), \gamma_2(t)) \le t\, d(\gamma_1(1), \gamma_2(1))$. //

— Balls are convex.
— Approximate midpoints close to midpoints.
— Contractible.

2016-04-20

Last time, Rauch and corollary:

If M, M_0 mflds, $K_M \le K_{M_0}$, ~~and~~ $m_0 \in M_0$, $r > 0$ s.t. $\exp_{m_0}|_{B_r(0)}$ is a diffeo, and $i : T_{m_0} M_0 \to T_m M$ is an isometry, ~~then~~ $f = \exp_m \circ i \circ \exp_{m_0}^{-1}$. Then f is length-increasing: $\forall$ curves γ in $B_r(m_0)$, $L(f(\gamma)) \ge L(\gamma)$.

Today, apply this: The CAT(0) condition

Def: A geodesic triangle is a loop made of three geodesic segments. Write $[p,q]$ for the edges, $\Delta(p,q,r)$ for the triangle.

If Δ is a geod. triangle, the comparison triangle $\bar\Delta = \bar\Delta(\bar p, \bar q, \bar r)$ is a triangle in $\mathbb{R}^2$ with the same side lengths.

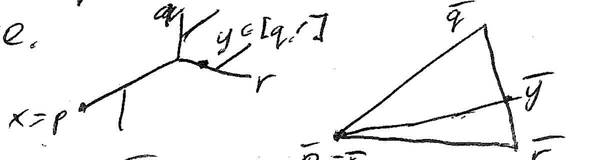
If $x \in [p,q]$, then comparison point $\bar x \in [\bar p, \bar q]$ is the point s.t. $d(x,p) = d(\bar x, \bar p)$.

(Def?) If X is a geodesic metric space, we say that X is CAT(0) if $\forall$ geodesic triangles Δ, $\forall x, y \in \Delta$, $d(x,y) \le d(\bar x, \bar y)$

– every measurement is bounded by the corresponding measurement in $\mathbb{R}^2$.

Note: No requirement that X is a manifold!

Ex: tree.



(but note that in general, we compare pts on edges too)

In general, checking CAT(0) is cumbersome, but:

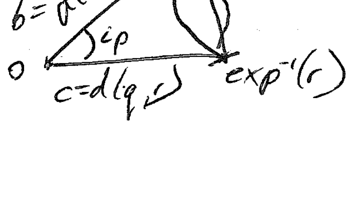
Thm: TFAE:

- X is CAT(0)
- $\forall \Delta(p,q,r)$, $\forall x \in [q,r]$, $d(p,x) \le d(\bar p, \bar x)$
- $\forall \Delta(p,q,r)$, if p, q, r distinct, then $\angle p \le \angle \bar p$

Pf omitted.

Thm: If M is a complete, simply-connected manifold and $K_M \le 0$, then M is CAT(0).

Pf: ~~By the corollary, if $m \in M$ then $\exp_m$~~ Let $\Delta = \Delta(p,q,r)$, $p \ne q \ne r$. By C-H, $\exp_p$ is a diffeo. By the corollary, $\exp_p$ is length-increasing. Consider $\exp_p^{-1}(\Delta)$:

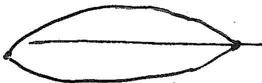


So $\sqrt{b^2 + c^2 - 2bc\cos\angle p} \le d(q,r) = \sqrt{b^2 + c^2 - 2bc\cos\angle \bar p}$

$\Rightarrow \angle p \le \angle \bar p$. //

Properties: If X is CAT(0), then:

- ~~$\forall x, y$, $\exists!$ geod $\gamma_{x,y}$ from x to y.~~

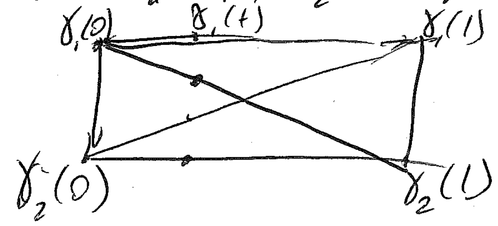


is a ~~geodesic triangle~~.

- ~~$\gamma_{x,y}$ varies continuously with its endpoints.~~

~~In fact,~~

- $d: X \times X \to \mathbb{R}$ is convex: Let γ_1, γ_2 be geodesics from p_i to q_i. Then $\forall t \in [0,1]$



Let λ be a geodesic from $\gamma_1(0)$ to $\gamma_2(1)$.

Then

$d(\gamma_1(t), \lambda(t)) \le d(\gamma_1(1), \gamma_2(1))\, t$

$d(\gamma_2(t), \lambda(t)) \le d(\gamma_1(0), \gamma_2(0))(1-t)$

by similar tris argument. So

$$d(\gamma_1(t), \gamma_2(t)) \le (1-t)\, d(\gamma_1(0), \gamma_2(0)) + t\, d(\gamma_1(1), \gamma_2(1))$$

- Cor: $\forall x, y \in X$, $\exists!$ $\gamma_{x,y}$ from x to y, which varies ~~smoothly~~ ctsly.

Pf: If γ_1, γ_2 ~~are geods, and $d(\gamma, 0$,~~ ~~are geods~~ are as above, then

$$d(\gamma_1(t), \gamma_2(t)) \le \max\{d(p_1, p_2), d(q_1, q_2)\}.$$

So if $p_1 = p_2$, $q_1 = q_2$, then $\gamma_1 = \gamma_2$.

Likewise, if $d(p_1, p_2), d(q_1, q_2) < \varepsilon$ then $d(\gamma_1, \gamma_2) < \varepsilon$.

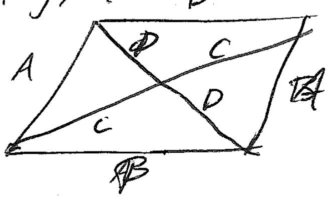
- Every metric ball is convex (easy)

- Every metric ball is strictly convex:

$\forall x \in X, \forall r, \varepsilon > 0$, $\exists \delta$ s.t. if $y, z \in B_x(r)$ and $d(y,z) > \varepsilon$, then $d(x, m) < r - \delta$,

where m is midpoint of y, z.

Pf: Parallelogram law.



$A^2 + B^2 = 2(C^2 + D^2)$

$\Rightarrow$ If $A, B \le r$, $D > \varepsilon$, then

$2C^2 + 2\varepsilon^2 \le 2r^2$

$\Rightarrow C \le \sqrt{r^2 - \varepsilon^2}$.

In fact, $d(x, m) < \cancel{\sqrt{\frac{d(x,y)^2 + d(x,z)^2}{2} - d(y,z)^2}}$ $r - \Omega\left(\frac{\varepsilon^2}{r}\right)$

~~This lets us define~~ Not true, e.g. in S^2, or in L^1-metric on $\mathbb{R}^2$. — ~~will let us prove some nice thms:~~

Standing: X CAT(0) complete

Thm: Suppose that $C \subset X$ is closed convex (~~if a geodesic~~ if $x, y \in C$, then $[x,y] \subset C$), X is complete (by reqs).

Then $\forall x \in X$, $\exists!$ $\pi(x) \in C$ s.t. $d(x, \pi(x)) = d(x, C)$

Further:

- $\forall x \notin C, y \in C$, if $y \ne \pi(x)$ then $\angle_{\pi(x)}(x, y) \ge \pi/2$.
- The map $x \mapsto \pi(x)$ is distance-decreasing.

Pf: Suppose that $y_1, \dots, \in C$, and $\lim_{n\to\infty} d(x, y_n) = d(x, C) = r$.

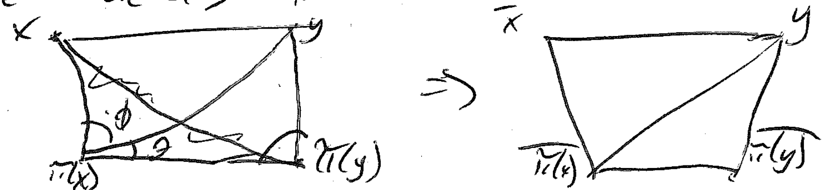
Claim: (y_i) is Cauchy. Suppose $y_i, y_j^{+} \in C$, $d(x, y_i) < r + \varepsilon$, $d(x, y_j^{+}) < r + \varepsilon$.

~~Then~~ Let m be the midpoint of y, y'. Then $m \in C$, so $r \le d(x, m) < r + \varepsilon - \Omega\left(\frac{d(y_i, y_j^{+})^2}{r}\right) \Rightarrow d(y_i, y_j^{+}) = O(\sqrt{r\varepsilon})$.

$\Rightarrow (y_i)$ is Cauchy. Further, the limit is ~~unig~~ well-defined — if (y_i') is another such sequence, then $y_1, y_1', y_2, y_2', \dots$ is ~~still~~ also Cauchy, so $\lim y_i = \lim y_i'$.

Angles: If $y \in C$, then $[y, \pi(x)] \subset C$. If $\angle_{\pi(x)}(x, y) < \pi/2$, then $d(x, [y, \pi(x)]) < d(x, \pi(x)) = d(x, C)$.

~~Ret~~ Distance-decreasing: Draw



Then $\angle \overline{\pi(y)} \ge \angle \pi(y) \ge \pi/2$

$\angle \overline{\pi(x)} \ge \theta + \phi \ge \angle \pi(x) \ge \pi/2$.

$\Rightarrow d(x, y) \ge d(\pi(x), \pi(y))$.

~~Another application of this corr~~ (and we'll see examples of convex elsewhere)

Thm (Cartan): If ~~f~~ X is complete, CAT(0), and $\Gamma \le \mathrm{Isom}(X)$ ~~is an isometry with~~ has a bounded orbit, then Γ fixes a point.

Pf:

Def: The radius of a set Y is the infimal radius of a ball containing Y —
$r_Y = \inf\{r \mid \exists x \in X,\ Y \subset B_r(x)\}$

Prop: If X is complete, CAT(0), $Y \subseteq X$ is a bounded set, then $\exists x_Y \in X$ s.t. $Y \subset B_{r_Y}(x_Y)$. (circumcenter)

Pf: Let $x_1, \dots \in X$ ~~be s.t.~~, $r_1, \dots > 0$ be s.t. $Y \subset B_{r_i}(x_i)$, $\lim_{i\to\infty} r_i = r_Y$.

Claim: ~~(x_i) is Cauchy.~~
- Same argument: if x, x' are s.t. $Y \subset B_{r_Y + \varepsilon}(x)$, $Y \subset B_{r_Y+\varepsilon}(x')$.

then $\forall y$, $d(x, y) \le r + \varepsilon$, $d(x', y) < r + \varepsilon$, so
$d(m, y) \le r + \varepsilon - \Omega\left(\frac{d(x, x')^2}{r_Y}\right)$

So, if ~~X is complete CA~~ Γ has a bounded orbit, let $Y = \Gamma x$, $c = c_Y$. Then Γ fixes Y, so it fixes c_Y by uniqueness. //

More generally, we can classify isoms of CAT(0) spaces similarly to those of hyp space.

Classification of isometries.

If $f: X \to X$ is an isometry, let $d_f : X \to \mathbb{R}$ be the displacement function $d_f(x) = d(x, f(x))$.

Let $|f| = \inf d_f$ = translation length of f,

let $Min(f) = d_f^{-1}(|f|)$.

Then:

- $Min(f)$ is invariant under f.
- $\forall \alpha \in Isom(X)$, $|f| = |\alpha \circ f \circ \alpha^{-1}|$, $Min(\alpha \circ f \circ \alpha^{-1}) = \alpha Min(f)$
- d_f is convex, so $Min(f)$ is closed, convex.

We say that either

- f is elliptic if f has a fixed pt ($|f| = 0$, $Min(f) \neq \phi$)
- f is hyperbolic if $|f| > 0$, $Min(f) \neq \phi$
- f is parabolic if $Min(f) = \phi$.

Ex: $X = \mathbb{R}^n$. $f(x) = Ax + b$. If f fixes a pt, then f is elliptic.

If f is not elliptic, then $x = Ax + b$ has no solutions $\Rightarrow$ $(A - I)$ is singular. Let $V = Null(A - I)$.

If $V = \mathbb{R}^n$, then f is a translation.

Otherwise, write $\mathbb{R}^n = V \oplus V^\perp$. A fixes $V, V^\perp$, so

$$f(v_1, v_2) = (v_1 + b_1, A_2 v_2 + b_2)$$

Further, $A_2 - I$ is nonsingular, so $f_2(v_2) = A_2 v_2 + b_2$ has a unique fixed pt.

~~Clearly~~, Then $d_f(v_1, v_2) \geq \|b_1\|$, with equality iff $f_2(v_2) = v_2$.

$\Rightarrow |f| = \|b_1\|$, $Min(f)$ is a plane parallel to V.

f acts by translation on $Min(f)$

(in particular, no parabolic isometries)

Ex: $X = H^n$: elliptic: d_f depends on distance to fixed pt.

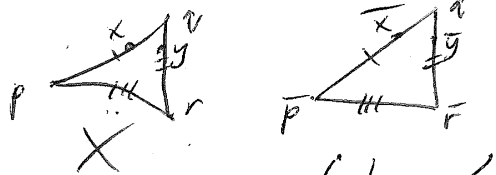
- min achieved at fixed pt

hyp: ~~always an axis~~

- $Min(f)$ = axis
- d_f depends on distance to axis, angle
- Next time, see why there's always ~~an axis~~ a unique axis.

2016-04-27

Last time: CAT(0) condition:



TFAE: $\forall x, y.\ d(x,y) \le d(\bar{x},\bar{y})$ $\quad \forall p,q,r,x,y$

– $\forall y \in [q,r],\ d(p,y) \le d(\bar{p},\bar{y})$ "

~~$\forall p,q,r$~~ $\angle p \le \angle \bar{p}$ "

(where $\angle_p(\gamma,\gamma') = \limsup_{t,t'\to 0} \bar{\angle}_p(\gamma(t),\gamma(t'))$) – for manifolds, this ~~seems~~ limit always exists.

Today symmetries, but first, a ~~lemma~~ ~~demonstrating how powerful this cond~~ is:

~~sec 2~~ Flat subspaces of CAT(0) spaces

Standing assumption: X is a CAT(0) space.

Flat Triangle Lemma: If $\Delta = \Delta(p,q,r)$, ~~and~~ $\bar{\Delta} = \Delta(\bar{p},\bar{q},\bar{r})$ is a comparison triangle, ~~then~~ and $\angle p = \angle \bar{p}$ or if $\exists x,y \in \Delta$ s.t. $d(x,y) = d(\bar{x},\bar{y})$ (and x,y not on same edge), then the convex hulls $C(\Delta)$ and $C(\bar{\Delta})$ are isometric (i.e., solid triangles)

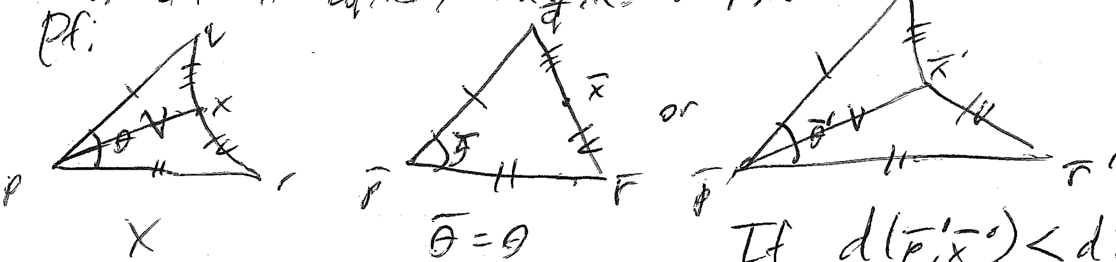
Note: $C(S)$ = minimal convex set containing S.

— in general, the convex hull of a triangle need not be 2-dimensional

Pf: Suppose $\angle p = \angle \bar{p}$

1. If $x \in [q,r]$, then $d(p,x) = d(\bar{p},\bar{x})$

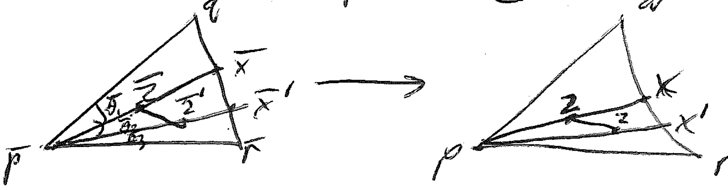
Pf:



If $d(\bar{p}',\bar{x}') < d(\bar{p},\bar{x})$, then $\bar{\theta}' < \bar{\theta}$ (by law of cosines)

But then $\theta \le \bar{\theta}' < \bar{\theta} = \theta$ $\times$. So $d(\bar{p},\bar{x}) = d(\bar{p}',\bar{x}') = d(p,x)$

Now, define a map $f: C(\bar{\Delta}) \to X$ by sending $[\bar{p},\bar{x}]$ to $[p,x]$ $\forall \bar{x} \in [\bar{q},\bar{r}]$.



Claim: Isometry. Suppose $\bar{z} \ne \bar{z}' \in \bar{\Delta}$. Claim: $d(z,z') = d(\bar{z},\bar{z}')$.

But all these lengths are equal, so $\bar{A}, \bar{B}, \bar{C}$ are comparison triangles for A, B, C. Then $\bar{\theta}_i \ge \theta_i \Rightarrow \bar{\theta}_i = \theta_i$. Important one is $\bar{\theta}_2 = \theta_2$.

If $d(z,z') < d(\bar{z},\bar{z}')$, then $\Delta(\bar{p},\bar{z},\bar{z}')$ has an angle smaller than $\bar{\theta}_2 \Rightarrow \theta_2 < \bar{\theta}_2$ $\times$.

Ex: Show that if $d(x,y) = d(\bar{x},\bar{y})$ nontrivially, then $\angle p = \angle \bar{p}$.

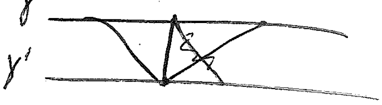
Similarly: Flat Quadrilateral Theorem: if p,q,r,s form a quadrilateral and ~~$\alpha+\beta+\gamma+\delta$~~ the sum of the angles is $\ge 2\pi$, then in fact, $C(p,q,r,s)$ is isometric to a Euclidean quad.

How does this affect the geometry?

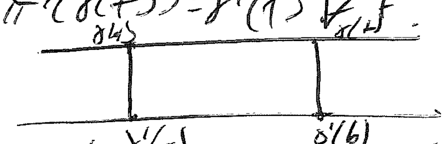
Def: We say that geods γ, γ' are parallel if $d(\gamma(t), \gamma'(t))$ is constant.

Flat Strip Theorem: Suppose $\gamma : \mathbb{R} \to X$, $\gamma' : \mathbb{R} \to X$ are geodesics s.t. $d(\gamma(t), \gamma'(t))$ is bounded. Then γ, γ' are parallel and $C(\gamma(\mathbb{R}) \cup \gamma'(\mathbb{R})) \cong I \times \mathbb{R}$.

Pf: - γ, γ' are parameterized with same speed.
Let $\pi : X \to \gamma$ be closest-point projection, parameterize so $\pi(\gamma'(0)) = \gamma(0)$.
Then $f(t) = d(\gamma(t), \gamma'(t))$ is convex, bdd $\Rightarrow$ constant.
In fact, $f_a(t) = d(\gamma(t), \gamma'(t+a))$ " " $\Rightarrow$ constant, and $f_a(0)$ is minimized when $a = 0$.



So $\pi(\gamma'(t)) = \gamma(t)$ $\forall t$. Likewise, if $\pi' : X \to \gamma'$, then $\pi'(\gamma(t)) = \gamma'(t)$ $\forall t$.
So: $\forall a, b \in \mathbb{R}$, draw:



All angles are $\geq \frac{\pi}{2}$, so this is a Euclidean rectangle.
So the map $I \times \mathbb{R} \longrightarrow X$
$\cancel{(t)} \; I \times t \longrightarrow [\gamma(t), \gamma'(t)]$ is an isom //

Product Decomp Thm: If γ is a geod, let
$$X_\gamma = \bigcup \{ \gamma'(\mathbb{R}) \mid \gamma' \text{ is parallel to } \gamma \}.$$
Then X_γ is convex, and $X_\gamma \cong X_\gamma^0 \times \mathbb{R}$, where X_γ^0 is a CAT(0)-space.
Pf: Exercise

break

Isometries: Recall: If $f \in \mathrm{Isom}(X)$
- $d_f(x) = d(x, f(x))$ — displacement function
- $|f| = \inf d_f$ — translation length
- $\mathrm{Min}(f) = d_f^{-1}(|f|)$.

Def: f is elliptic $\Leftrightarrow$ $|f| = 0$, $\mathrm{Min}(f) \neq \emptyset$
f is hyperbolic $\Leftrightarrow$ $|f| > 0$, $\mathrm{Min}(f) \neq \emptyset$
f is parabolic $\Leftrightarrow$ $\mathrm{Min}(f) = \emptyset$.

Ex: In H^n, this ~~case~~ is the same as the previous def:
elliptic: fix a point $\Rightarrow$ $\mathrm{Min}(f)$ is totally geodesic
hyperbolic: fix exactly two points in $\partial_\infty H^n \Rightarrow |f| > 0$, $\mathrm{Min}(f) =$ geodesic (axes)
Parabolic: fix exactly one point in $\partial_\infty H^n \Rightarrow |f| = 0$, $\mathrm{Min}(f) = \emptyset$.
Exercise: All isoms of $\mathbb{R}^n$ are elliptic or hyperbolic.
If f is hyperbolic, there is a decomp $\mathbb{R}^n = \overset{\mathrm{Min}(f)}{V} \oplus V^\perp$
where f acts on V by translation, $V^\perp$ by rotation/reflection, ~~Min(f) = V~~.

Generally: If $f \in \underline{\mathrm{Isom}}(X)$, an $\underline{\text{axis}}$ of f is a geodesic γ s.t. $\forall t$, $f(\gamma(t)) = \gamma(t+t_a)$

Lem: Any axis is contained in Min(f) by projection + translation by $|f|$.

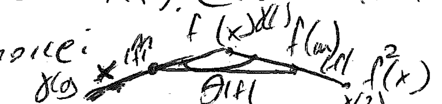
Thm: ~~If $f \in \mathrm{Isom}(X)$, an axis of f is a geodesic γ s.t. $\forall t$, $f($~~

If $f \in \mathrm{Isom}(X)$ is hyperbolic, then f has ~~an~~ axis. In fact,

- $\mathrm{Min}(f) = \bigcup\{\text{axes of } f\}$ and all axes are parallel.
- $\mathrm{Min}(f) \cong Y \times \mathbb{R}$, where Y is CAT(0), and $f(y,t) = (y, t + |f|)$
- If g commutes with f, then $g(y,t) = (g_1(y), t+b)$, where $b \in \mathbb{R}$, $g_1 \in \mathrm{Isom}(Y)$.

Pf: Recall: Min(f) is convex, $f(\mathrm{Min}(f)) = \mathrm{Min}(f)$, more generally, if $[f,g] = id$, then $g(\mathrm{Min}(f)) = \mathrm{Min}(f)$.

Suppose $x \in \mathrm{Min}(f)$. Claim: x lies on an axis.

Natural choice: 

~~Def~~ Let γ be the broken geodesic connecting $f^{-1}(x)$, x, $f(x)$, ...

Claim: γ is a geodesic. ~~i.e. $\theta = \pi$~~ Then $\gamma|_{[0,2]}$ is a geod.

Let $m = \mathrm{mid}(x, f(x))$. Then ~~$d_f(m) = |f|$~~, $m \in \mathrm{Min}(f)$, so ~~$d_f(m)$~~

$d(m, f(m)) = |f|$. So ~~the broken geod γ is a geodesic~~

But γ is a path of length $|f|$ from m to $f(m)$ $\Rightarrow$ ~~γ is locally a geodesic.~~

~~γ is ... a geodesic from m to $f(m$~~ ~~$\theta = \pi$~~ ~~$\theta = \pi$~~.

~~$[m, f(m)] \in \gamma$~~. $\Rightarrow$ ~~γ is a geodesic~~

~~$[m, f(m)]$~~ γ ~~is a geo~~ $\gamma_{[\frac{1}{2}, \frac{3}{2}]}$ is a geod. $\Rightarrow$ γ is a local geodesic

Ex: Every local geodesic in a CAT(0) space is a geodesic. (induction)

~~Claim: θ ... γ~~

- All axes are parallel; ~~contained~~ minset.

Suppose γ, γ' are axes. Then $\underbrace{d(\gamma(t), \gamma'(t))}_{g(t)} = d(f(\gamma(t)), f(\gamma'(t)))$

Parameterize ~~so $f(\gamma(t)) \in \gamma$~~ unit-speed

so $f(\gamma(t)) = \gamma(t+1)$. $\quad = d(\gamma(t+1), \gamma'(t+1))$

So $g(t)$ is periodic, convex $\Rightarrow$ constant.

$\Rightarrow \gamma, \gamma'$ are parallel.

(~~and f translates both by same distance~~)

- $\mathrm{Min}(f) \cong Y \times \mathbb{R}$: Let γ be an axis, $M = \mathrm{Min}(f)$.

Then M is convex $\Rightarrow M$ is CAT(0). ~~So M_γ~~ We know $M = \bigcup\{\text{axes}\}$,

~~and M_γ ... geods parallel to $\gamma\} = \mathbb{R}^2$~~. ~~So $M \cong Y \times \mathbb{R}$~~

~~Further, f translates each axis by $|f|$, so $f(y,t) = (y, t+|f|)$~~

~~If λ is a geod~~ Let ~~γ be~~ an axis. ~~Then~~ If $\lambda \subset M$,

~~$\lambda \parallel \gamma$, then~~ $\gamma \parallel \lambda$, ~~let γ' be the axis through $\lambda(0)$~~. Then

$\gamma \parallel \gamma'$ ~~$\Rightarrow \gamma' \parallel \lambda \Rightarrow \gamma' = \lambda$~~.

~~So~~ so M is a CAT(0) space that's the union of parallel geods

Product Decomp Theorem $\Rightarrow M \cong Y \times \mathbb{R}$.

- ~~If g commutes with f, then: $g(M)$ Then $g(M)=M$~~
- Suppose g commutes with f, γ an axis of f:
Then:
  - $f(\gamma(t)) = \gamma(t+|f|)$ so
  $$f(g(\gamma(t))) = g\circ f(\gamma(t)) = g(\gamma(t+|f|))$$
  So $g\circ\gamma$ is an axis
  - $\Rightarrow$ $g(M)=M$, g sends axes to axes,
  $$\Rightarrow \quad g(y,t) = (g_1(y), t+b)$$
  (g action sets of axes by an isom, g acts on each axis by an isom.) //

~~Preview of next time:~~ And this is a powerful result:
~~One good~~ Consider a basic Question: If M is a ~~compact~~ manifold, does it have a metric with $K_M \le 0$?

If M compact, $K_M \le 0$, then $X = \tilde{M}$ is $CAT(0)$, and $\Gamma = \pi_1(M)$ acts on X by isometries.
In fact, hyperbolic isometries. ~~This is a strong condition:~~
~~We can prove a splitting theorem.~~ We can use the axes to prove a splitting lemma:

Thm: If $\Gamma \subset \mathrm{Isom}(X)$ is finitely generated, ~~acts on X~~, and if $a \in Z(\Gamma)$ is a hyperbolic isometry, then $\exists$ a finite-index $G \subset \Gamma$ s.t. $G \cong \langle a\rangle \times H$

Pf: By theorem, $\mathrm{Min}(a) \cong Y \times \mathbb{R}$. If $g\in\Gamma$, then g acts on $\mathrm{Min}(a)$,
$$g(y,t) = (g_1(y), t+b_g).$$
Let $\beta : \Gamma \to \mathbb{R}$
$\beta(g) = b_g$. Then $\beta(a) \neq 0$, and since Γ is f.g,
$\beta(\Gamma) \cong \mathbb{Z}^n$

$\exists\, \rho : \mathbb{Z}^n \to \mathbb{Z}$ s.t. $\rho(\beta(a)) \neq 0$. Let $\phi = \rho\circ\beta : \Gamma \to \mathbb{Z}$
Let $G = \phi^{-1}(\phi(a)\mathbb{Z})$
$H = \ker \phi$.

Then $H\langle a\rangle = G$,
$H \cap \langle a\rangle = 0$,
and $a \in Z(H)$, so
$G \cong \langle a\rangle \times H$. //