This is Diff Geo II.

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 Bring paper for email list.
 Office hrs: (Catterall) Monday, 2-3?

Jeff said you covered: metrics, connections, Levi-Civita connection, curvature tensor.

(Do you know how to calculate length of curves? Suppose I give you a surface in \(\mathbb{R}^3\).
 - describe a connection in a coordinate patch? say a torus.
 - what is uniqueness? Calculate the Levi-Civita connection."
 - calculate others.
What can you calculate?)

What I want to do this term:
These are essentially all local. Jeff said that he showed you how to use curvature to calculate Taylor expansions of metric, right?
- this is local structure. This term: continue studying local structure, how local set but also focus on how local structure determines global structure. If you know something about curvature (pos, curv, neg, curv...)
that, etc.) But does that say about the global structure of the manifold?

Let's start by reviewing some of the basics. I like to work with at least one motivating example, with submanifolds of \(\mathbb{R}^n\).

Nash showed that every closed manifold embeds smoothly, isometrically in \(\mathbb{R}^n\) for large enough \(n\), so this is not much of a restriction.

Q: How can we find shortest paths in \(\mathbb{R}^n\)?

Calculus of variations: Let \(Y: [0, 1] \to \mathbb{R}^n\) be a smooth path, define:
\[
E_c^b(Y) = \int_0^1 \sqrt{g_{ab}(\dot{Y}^a(\tau), \dot{Y}^b(\tau))} d\tau
\]

By Cauchy-Schwarz, \[
\int_0^1 \sqrt{\frac{dx}{dt}} \sqrt{\frac{dt}{dx}} = \int_0^1 \sqrt{\sqrt{g_{ab}(\dot{Y}^a(\tau), \dot{Y}^b(\tau))}} d\tau = \int_0^1 \sqrt{\sqrt{g_{ab}(\dot{Y}^a(\tau), \dot{Y}^b(\tau))}} d\tau
\]

So, \(L(Y) = \sqrt{\int_0^1 d\tau} = \sqrt{\int_0^1 \sqrt{\sqrt{g_{ab}(\dot{Y}^a(\tau), \dot{Y}^b(\tau))}} d\tau} = \sqrt{n - a E_c^b(Y)}
\]

E is minimized when \(Y\) is minimal length, constant speed.

How do variations affect \(E\)? Let \(\delta E = E'_t\)

Let \(h_t: [0, 1] \to \mathbb{R}^n\) be a smooth family of \(Y\)

What is \[
\frac{d}{dt} E(h_t) = \frac{1}{2} \int_0^1 \sqrt{\delta E^2 (h_t)} dt \frac{d}{dt} h_t + \int_0^1 \frac{1}{2} \sqrt{\delta E^2 (h_t)} dt \frac{d}{dt} h_t
\]
\[ \frac{d}{dt} \left( \frac{\partial}{\partial u} \right) \approx \frac{\partial^2}{\partial x \partial t} + \frac{\partial}{\partial y} \left( \frac{\partial^2}{\partial y^2} \right) \]

Let \( U(\cdot, t), V(\cdot, t), A(\cdot, t) \) be...

edge term

If \( h \) is a shortest path in \( M \) from \( p \) to \( q \), then \( \frac{\partial^2 h}{\partial t^2} = 0 \) - zero accel.

If \( K \) is a shortest path in \( M \) from \( p \) to \( q \), then \( \frac{\partial^2 h}{\partial t^2} \) is normal to \( M \), no acceleration tangent to \( M \).

How do we apply this in general? We need a notion of acceleration tangent to \( M \). 

\( M \subset \mathbb{R}^n \), easy, \( \Phi \) is a vector field on \( Y \) (i.e., \( \Phi \cdot V(\cdot, t) \)).

we let \( \nabla Y \) be the projection of \( \Phi \) onto \( M \).

Goal: If \( \Phi \) is a shortest path, then \( \nabla Y = 0 \) (zero accel).

But why is the invariant? Not obvious, invariant. To prove invariance, we generalize to ... track?

Connections: \( \nabla: T_M \times \mathcal{V}(M) \rightarrow T_M \) for vector field on \( M \).

If \( X \in T_M, Y \in \mathcal{V}(M) \), \( \nabla_Y X \in T_M \) is the covariant derivative of \( Y \) in direction \( X \).

\( \nabla_X Y \) is bilinear in \( X, Y \)

a covariant derivative - i.e., \( \nabla_X Y = X(p)Y_{p} + A(p)\nabla X_{p} Y \)

- varies smoothly - i.e., \( \nabla: \mathcal{V}(M) \times \mathcal{V}(M) \rightarrow \mathcal{V}(M) \)

(Ex: Directional derivative in \( \mathbb{R}^n \)).
But in general, many connections—

if \( u_1, \ldots, u_n \) are coordinate tangents on a patch, and corresponding \( v \) fields \( \partial_1, \ldots, \partial_k \), then the connection is determined by the fields

\[ \nabla_i \partial_j = \sum \Gamma_{ij}^k \partial_k \]

(Christoffel symbols of connection), so \( \nabla \) is determined by \( n^3 \) smooth \( \Gamma \)s.

Entirely locally.

If \( M \subset \mathbb{R}^n \) we define the \textit{tangential connection} \( \nabla^T \) by

let \( X, Y \) be smooth extensions of \( \partial_i, \partial_j \) let \( p^T : M \times \mathbb{R}^n \to \mathbb{T} M \) be the projection, define

\[ \nabla^T Y = p^T (\nabla\partial_j Y) \]

where \( \nabla \) is Euclidean (in directional derivative). This is well-defined.

We can prove one of the basic facts of connections is derive along curves:

Thus if \( \nabla \) is a connection \( \exists ! D^1 : \mathcal{V}(U) \to \mathcal{V}(U) \) (derivative along \( \gamma \))

s.t.

\[ D^1 (V + W) = D^1 (V) + D^1 (W) \]

\[ D^1 (fV) = \frac{df}{dt} \partial_{\gamma(t)} + f D^1 (V) \]

\[ D^1 \partial_\gamma = D^1 (\partial_t) \]

If \( V \) is the restriction of \( \mathbf{v} \in \mathcal{V}(M) \), then \( D^1 V = \frac{d}{dt} \mathbf{v} |_{t=0} \).

So we can rewrite good eq. as \( \frac{dV}{dt} = 0 \)

Equivalently, we say that the velocity field \( \frac{d}{dt} \) is parallel.

Then if \( \gamma \subset T_{x_0} M \), then \( \exists ! \) parallel \( v \) field on \( U \) s.t. \( V(t_0) = \mathbf{v} \).

We define if \( \gamma \) is a curve \( \subset T_{x_0} M \), we define parallel transport

\[ P_{t_0, t_1}^\gamma : T_{x_0} M \to T_{x(t_1)} M \quad \text{so that} \quad P_{t_0, t_1}^\gamma e_{\gamma(t_0)} = V(t_1) \]

Thus far nothing we've done used the metric on \( M \) (well except when \( M \) is a subbundle) - connections, parallel transport, etc. are all purely smooth notions. Let's start bringing in the metric - that's all where uniqueness will come in.

The tangential connection is compatible with the induced metric on \( M \).

That is, it satisfies the following equiv equations:

- If \( V, W \) are parallel \( v \) fields on \( X \), then \( \langle V, W \rangle \) is constant.

\[ \nabla^T \partial_\gamma = \frac{d}{dt} \partial_{\gamma(t)} \]

- If \( U, V \in \mathcal{V}(X) \), then \( \langle V, W \rangle = \langle D^1 U, V \rangle \langle U, D^1 W \rangle \)

If \( X, Y, Z \in \mathcal{V}(X) \), then \( X \langle Y, Z \rangle = \langle D^1 X, Y \rangle \langle Y, Z \rangle + \langle Y, D^1 Z \rangle \).

Note:

- Unique? No: if \( X, Y, Z \in \mathcal{V}(X) \), then we get \( n^3 \) linear equations but \( X, Y, Z \) give one eq, so we have \( n^3 - 2 \) compatible connections.
For any not to construct “twisted” connections \( F : \mathbb{R}^n \to SO_n \), so that \( P \) \( X \times \mathbb{R} \) \( F(\theta(0)), F(\theta(0')) \). 

Define \( \nabla \) so that the sections \( \{ p \cdot F(\theta(0)) \mid p \in \mathbb{R}^n \} \) are parallel. This is compatible with metric, i.e. \( P \) \( \phi \) \( \psi \) \( \mathbb{R}^n \). 

What distinguishes these is torsion. 

Then the tangential connection is torsion-free, i.e. 

\( \mathbb{D}(X, y) = -\nabla_X Y - \nabla_Y X - [X, Y] e^\theta(0, 1) \mathbb{R}^n \). 

If \( M \) is a Riemannian manifold, \( \mathbb{D} \) is connection that is torsion-free, compatible with metric. 

Then, dimension constant, \( \mathbb{D} \) is antisymmetric, so \( \mathbb{D}(\cdot, \cdot, \cdot) = 0 \) leads to scalar equation, \( \frac{m^2}{2} \langle \psi \rangle \). 

Combining these, with the \( m \) \( \frac{2}{m} \) \( \psi \) \( \text{compatability} \). 

We get uniqueness and a formula. 

\( \langle \mathbb{D} \rangle \), \( \langle \psi \rangle \). 

\( \frac{1}{2} \langle \mathbb{D} \rangle \), \( \langle \psi \rangle \). 

Let \( x^i \) be coordinate func, \( \delta_i = \frac{\partial}{\partial x^i} \) be vectors, \( g_{ij} = \langle \delta_i, \delta_j \rangle \). \( \mathbb{R}^n \) be st. 

\( \nabla_{\delta_i} \delta_j = \sum_k g_{ik} \delta_k \). Then, \( \mathbb{D} \langle \mathbb{D} \rangle = \langle g_{ij} \rangle \). 

\( g_{ij} \sum_k (\delta_k g_{ik} + \delta_i g_{kj}) \). 

Curvature of a connection: 

Define \( \mathbb{R}(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \). 

Then, \( \mathbb{R} \) is linear in \( X, Y, Z \) (in fact precise formula). 

\( \mathbb{R} \) measures the path-dependence of parallel transport. 

Ex: sphere, saddle? 

\( \mathbb{R} \) satisfies: 

\( \mathbb{R}(X, Y) Z + \mathbb{R}(X, Y) Z = 0 \). 

\( \mathbb{R}(X, Y) Z + \mathbb{R}(Y, Z) X + \mathbb{R}(Z, X) Y = 0 \). 

\( \langle \mathbb{R}(X, Y) Z, W \rangle + \langle \mathbb{R}(X, Y) W, Z \rangle = 0 \). 

\( \langle \mathbb{R}(X, Y) Z, W \rangle = \langle \mathbb{R}(Z, W) X, Y \rangle \).
Generally, full curvature tensor is too much information to work with conveniently; so we'll define various slices, averages, contractions.
- For surfaces, all the identities mean that the tensor boils down to one number: Gaussian curvature.
- In higher dimensions, various invariants derived from tensor:
  Ricci curvature, sectional, scalar, curv.

Next time: geodesics: when do shortest paths exist, how to find them, etc.

Last time: connections, parallel transport.

Today: curvature, geodesics.

Let $M$ be a manifold. \( \nabla \) the Riemannian connection on $M$.

If $X, Y, Z \in \mathfrak{X}(M)$ define $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - [X, Y]Z$

Then:
- $R$ is "trilinear in $X, Y, Z$.
- $R$ is a tensor.
- $R$ satisfies various symmetries:
  $$R(X, Y)Z + R(Y, X)Z = 0$$
  $$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$
  $$\langle R(X, Y)Z, W \rangle + \langle R(Y, W)X, Z \rangle + \langle R(Z, X)Y, W \rangle = 0$$
  $$\langle R(X, Y)Z, W \rangle = \langle R(Z, W)X, Y \rangle$$

- $R$ measures the path-dependence of parallel transport.

- If $\gamma$ is a path from $p$ to $q$, then $R_p: T_pM \to T_qM$ is parallel transport. If $\gamma$ is a closed path in $M$, then $R_p$ is a map from $T_pM$ to itself, called the holonomy of $\gamma$.

If $p = q$ for every null-homotopy closed curve $\gamma$, then $R_p = 0$, and $M$ is flat (as it has to be). Then $R = 0$.

Then:
If $X, Y \in \mathfrak{X}(M)$ and if $f: \mathbb{R}^2 \to M$ is a map s.t. $f(0, 0) = x, \frac{\partial f}{\partial x} = X, \frac{\partial f}{\partial y} = Y$, then

$$R(X, Y)Z = \lim_{\epsilon \to 0} \frac{2 - P^\epsilon(Z)}{\epsilon^2},$$

where $X_\epsilon$ is the boundary of the $\epsilon$-square in $\mathbb{R}^2$.

So we can calculate $R$. This is like Stokes' theorem.

This is like Stokes' theorem - given a curve $\gamma$ and a surface, we can calculate parallel transport by integrating curvature equations along the integration around the boundary or by integrating the curvature on the surface.
Ex: If \( M \) is a 2-manifold, the symmetries reduce \( M \) to a 2-form, denoted \( \gamma \). A 2-form is characterized by the area form \( \gamma(X,Y) = \langle R(X,Y)Y, X \rangle \), so its curvature is characterized by the ratio of the 2-form to the area form. We define the Gaussian curvature of \( M \) by

\[
K = \frac{\langle R(X,Y)Y, X \rangle}{\langle X, Y \rangle^2}
\]

If \( M \) is the unit sphere, we can calculate using spherical geometry:

**Fact:** If \( A, B, C, \) are vertices of a triangle \( \triangle ABC \) on \( S^2 \), then area \( \triangle ABC = \pi + \alpha + \beta + \gamma \).

**What is the holonomy?**

- Construct a parallelogram \( AB'C_1C \) (tilted relative to \( ABC \))
- Start with direction of the geodesic

\[
P_{\Delta AB} = \text{rotation by } \tau = \alpha + \pi - \beta + \gamma - \pi = \text{counter clockwise}
\]

So, if \( X, Y \) are orthogonal, then

\[
\langle X, Y \rangle = 0 \quad \Rightarrow \quad P_{\Delta AB} = \text{counter clockwise}
\]

In general, \( P_{\Delta AB} = \text{counter clockwise} \).

\[
\Rightarrow K = 1
\]

In general, if \( \mathbb{S}^2 \) has \( M \) as \( \mathbb{S}^2(r) \), then \( K = \frac{1}{r^2} \).

[This is not too far from the Gauss-Bonnet formula, some details omitted]

**Theorem:** For Every Geodesic Triangle on a General Surface, we have

\[
\int K dA = \alpha + \beta + \gamma - \pi
\]

By usual Stokes arguments, if \( U \subset M \), then \( P_{\Delta AB} = \text{counter clockwise} \).

So, if you have \( \text{OTM} \), it is a geodesic triangle, with angles \( \alpha, \beta, \gamma \).

\[
P_{\Delta AB} = \text{counter clockwise} \quad \Rightarrow \quad \alpha + \beta + \gamma - \pi = \text{counter clockwise}
\]

So, if \( M \) is a surface with a geodesic triangle, then

\[
\Rightarrow \quad \text{for every geodesic triangle in } M, \quad \int K dA = \alpha + \beta + \gamma - \pi
\]

So, if \( M \) has a geodesic triangulation with \( \Delta \) vertices, \( E \) edges, \( F \) faces, \( \epsilon \) we have

\[
\sum_{\Delta} K dA = \sum_{\Delta} \left( \alpha + \beta + \gamma - \pi \right)
\]

\[
= 2\pi V - \pi F - 2\pi E + 2\pi (V - E + F)
\]

\[
= 2\pi K(M)
\]

**Ex:** Another characterization of \( R \): it's the deviation from being flat.

We say \( M \) is flat if \( R = 0 \) — what does this imply?
Prop: If $M$ is flat and $\Sigma$ a null-totally-geodesic closed curve in $M$, then $p^\flat = \text{id}$. 

Pf.: *Stokes* Theorem. Consider a square $\Sigma$. 

Thm: If $M$ is flat, then $M$ is locally isometric to $\mathbb{R}^n$. 

Lem: If $M$ is flat, then $M$ locally has a field of parallel orthonormal frames. 

Pf. Lemma: If $M$ is flat and $\gamma$ is a null-totally-geodesic closed curve in $M$, then $p^\flat = \text{id}$. 

Pf. Suppose $I = [0,1]$, $\alpha: I^2 \to M$ be st. $\alpha(0^2) = \gamma$. Let $\nu \in T \alpha(\gamma)$ and let $X \in \mathfrak{X}(\alpha)$ be the field s.t.

$L_X(\gamma) = \nu$ and $X$ is parallel along $\gamma$. 

Claim: $\nabla_{\nu}^X X = 0$ where $\nu_i = \frac{\partial \alpha}{\partial x^i}$.

Weil: $\nabla_{\nu}^X = \nabla_{\nu_1}^X = \nabla_{\nu_2}^X = \nabla_{\nu_3}^X = \nabla_{\nu_4}^X = 0, \nabla_{\nu_5}^X = \nabla_{\nu_6}^X = \nabla_{\nu_7}^X = \nabla_{\nu_8}^X = 0 \quad (\text{since } \nabla_{\nu}^X X \equiv 0) \quad (\text{compatibility}) \quad (\text{flatness})$

So $\nabla_{\nu}^X X = 0 \Rightarrow X$ is parallel $\Rightarrow p^\flat = \text{id}$. 

Cor: $M$ locally has a field of parallel orthonormal frames if $\forall u \in M \exists \text{ a field of parallel orthonormal frames } \{X_1, \ldots, X_m\}$ s.t. $X_i$ are parallel, orthonormal. 

Pf.: Let $v, \omega \in T \gamma$ be orthonormal, let $U \ni x$ be a ball, define $U \ni y$ be s.t. $v = \mathfrak{X}(\gamma)$, $\forall x \in U$, let $y$ be a path from $x$ to $y$, define $X_i(y) = p_y(v_i)$. There exists a well-defined parallel, orthonormal.

Finally, consider $[X_i, X_j]$. Since $\nabla$ is torsion-free,

$\nabla_{X_i} X_j = \nabla_{X_j} X_i - \nabla_{X_i} X_j = 0.$

These are compatible v.f. fields so there exists a map $\beta: U \subseteq \mathbb{R}^m \to U$ s.t. $\beta(0) = x_0$, and $\frac{d}{dx} \beta = X_i \forall i.$ by Frobenius' Theorem. 

So flat $\Rightarrow$ no holonomy $\Rightarrow$ locally isometric.
Geodesics: \( \gamma: I \rightarrow M \) is a geodesic if its velocity field \( \dot{\gamma} \) is parallel, i.e., \( D\frac{\dot{\gamma}}{dt} = 0 \). (In particular, \( \gamma \) has constant speed.

Then: (Assumption of basic facts, proven in last term)
- \( \forall \gamma \in M, \forall \nu \in T\gamma \mathcal{M}, \exists \tau > 0 \) s.t. \( (\varepsilon, \varepsilon) \rightarrow M \)
  - a geod. s.t. \( \gamma(0) = \gamma, \gamma'(0) = \nu \)
  - local existence

Furthermore, we assert that if \( \gamma \) is a geod.
- \( \forall \gamma \in M, \exists \text{ a ball } \mathcal{B}(\gamma, R) \exists \varepsilon > 0 \) s.t.
  - \( \forall \nu \in \mathcal{B}(\gamma, R), \forall \varepsilon > 0 \) s.t.
    - \( \gamma^\nu \in T\gamma \mathcal{M}, \forall \varepsilon > 0 \) s.t.
      - \( \gamma \cdot \gamma^\nu \) is a unique geod.
      - \( \gamma : (-\varepsilon, \varepsilon) \rightarrow M \) s.t.
        - \( \gamma(0) = \gamma, \gamma'(0) = \nu \), \( \gamma'(0) = \nu \)

It we define \( \exp(\nu) = \gamma(0) \), then \( \exp : WCT \gamma \mathcal{M} \rightarrow M \) is a smooth map. Define a submersion \( \mathcal{M} \rightarrow WCT \gamma \mathcal{M} \).

- \( \exp : WCT \gamma \mathcal{M} \rightarrow M \) is a submersion. Define a submersion \( \mathcal{M} \rightarrow WCT \gamma \mathcal{M} \).
  - Furthermore, if you fix \( p \) and let \( \nu \) vary, this is locally invertible (Inverse of F.F.D.

This depends smoothly on \( \gamma, \nu \).

- \( \forall \nu \in W, \forall p \in W \) is mapped diffeomorphically.
  - These exist locally. Not hard to show following:

Thus: Further, you can use F.D.F. to prove follows:
- \( \forall \gamma \in \mathcal{M}, \exists \nu \) such that \( \gamma^\nu \) is a unique geod. of length \( \gamma \).
  - this geod. depends smoothly on \( \gamma, \nu \)
  - \( \forall \nu \in \mathcal{W} \), the map \( \exp \) sends \( \mathcal{W} \) diffeomorphically to \( \gamma \).

- \( \gamma \) is a geod.

- \( \gamma \) is a geod.

What makes these interesting is that they are related to minimal curves.

Jeff: And you see Gauss's Lemma? What version?

Optic Lemma: If \( s \) is as above, \( r < \varepsilon \), then \( \gamma_s \) is orthogonal to \( \exp(\gamma(0)) \).

Lemma: Any \( \gamma \) path \( \gamma \) to \( \gamma \) is a geod. of length \( \ell \) with both
- \( \exp \) s.t. to \( \exp \gamma(0) \) for \( \gamma \) length \( \ell \) with both
- \( \exp \) s.t. to \( \exp \gamma(0) \) for \( \gamma \) length \( \ell \) with both
Cor: Any length-minimizing path is a geodesic.
(called a minimal geodesic)

If $\gamma$ is length-minimizing, consider $p \in \gamma$, and a neighborhood $V$ of $p$ in the lemma. Then shortest paths Pick $q$ on one side, $r$ on other.

- Cursive copy?