STUDY GUIDE

This is a guide to some of the skills, examples, and theorems that we’ve covered in class.

1. **Skills**

- **Groups**
  - Define *group*
  - Define *abelian*
  - Decide whether a given set and operation form a group
  - Compute using group properties
  - Construct groups of symmetries
  - Calculate with multiplication tables
  - Prove facts about groups

- **Subgroups**
  - Define *subgroup*
  - Define *cyclic group*
  - Determine whether a group is cyclic
  - Identify subgroups
  - Find subgroups of a group
  - Prove facts about subgroups

- **Cosets**
  - Define *coset*
  - Define the *index* and *order* of a subgroup
  - Define the *order of an element*
  - Calculate the cosets of a subgroup
  - Calculate the index of a subgroup
  - State Lagrange’s Theorem
  - Reason using Lagrange’s Theorem
  - Prove facts about cosets
  - Use counting arguments to measure the order or index of a subgroup
  - Use counting arguments to count products

- **Normal subgroups**
  - Define *normal subgroup*
  - Identify normal subgroups
  - Find normal subgroups of a group
  - Prove facts about normal subgroups

- **Quotients**
  - Define *quotient group*
  - Calculate $G/N$
  - Calculate with quotient groups
  - Use the Correspondence Theorem (2.7.5)
  - Prove facts about quotients

- **Homomorphisms**
  - Define *homomorphism* and *isomorphism*
  - Identify homomorphisms and isomorphisms

*Date: November 17, 2014.*
– Check whether functions are well-defined
– Use properties of homomorphisms
– Calculate the image/kernel of a homomorphism
– Calculate the homomorphisms from one group to another
– Use the first isomorphism theorem to construct isomorphisms between quotients and images
– Prove facts about homomorphisms
– Determine whether two groups are isomorphic
• Automorphisms and automorphism groups
  – Define automorphism
  – Calculate Aut(G)
  – Prove facts about automorphisms and automorphism groups
• Products
  – Define direct product
  – Define semidirect product
  – Construct direct and semidirect products.
  – Prove facts about products
• Permutation groups
  – Define symmetric group
  – Define the signature of a permutation
  – Calculate cycle decompositions of permutations
  – Calculate products of permutations
• Finite abelian groups
  – Classify the finite abelian groups of a given order
  – Determine when two finite abelian groups are isomorphic
  – Find subgroups, quotients, and homomorphisms of finite abelian groups
• Rings
  – Define ring, commutative ring, unital ring, integral domain, division ring, and field
  – Decide whether a set with given operations is a ring, commutative ring, unital ring, etc.
  – Compute using ring properties
  – Define the characteristic of a ring
  – Prove facts about rings
• Homomorphisms
  – Define ring homomorphism
  – Determine whether there are homomorphisms between two rings
  – Prove whether two rings are isomorphic
  – Compute the kernel of a homomorphism
  – Prove facts about homomorphisms
• Ideals and quotients
  – Define ideal
  – Define quotient ring
  – Decide whether a given subset of a ring is an ideal
  – Construct and calculate with quotients
  – Prove facts about ideals and quotients

2. Important examples

2.1. Groups. What are some subgroups, normal subgroups, quotients, etc. of these groups? What are some homomorphisms between them?

• Cyclic groups: \( \mathbb{Z}_n \)
• Systems of numbers: $(\mathbb{Z}, +), (\mathbb{R}, +), (\mathbb{R}^*, \times), (\mathbb{C}^*, \times), (\mathbb{Q}, +), (\mathbb{Q}^*, \times)$
• Vector spaces: $(\mathbb{R}^n, +)$
• Groups of symmetries: $\text{Sym}(\Delta)$, etc.
• Matrix groups: $\text{GL}_n(\mathbb{R}), \text{SL}_n(\mathbb{R})$
• Affine groups: $A_1$
• Direct products: $A \times B$
• Semidirect products: $A \ltimes B$
• Dihedral groups: $D_{2n} \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_n$
• Permutation groups

2.2. **Rings.** What are some ideals and quotients of these rings? Are they commutative rings, unital rings, integral domains, fields, etc? What elements are invertible? What are some homomorphisms between them?

- Systems of numbers: $\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{H}$
- Integers mod $n$: $\mathbb{Z}_n$
- Polynomials: $\mathbb{Z}[t], \mathbb{R}[t], \mathbb{Z}_n[t]$
- Matrices: $M_n(\mathbb{R})$
- Functions: $C([0,1]) = \{ f : [0,1] \to \mathbb{R} \mid f \text{ is continuous} \}$
- Square roots: $\mathbb{Z}[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Z} \}$, $\mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \}$

3. **Important theorems**

Can you state these theorems? Can you use them to solve problems?

- Lagrange’s Theorem
- Counting products (2.5.1)
- Characterizing normality (Sec. 2.6)
- First Isomorphism Theorem (2.7.1, see also 11.3 in Judson)
- Cayley’s Theorem
- Cycle decompositions (2.10.1)
- Signature of a permutation (2.10)
- Fundamental Theorem of Finite Abelian Groups (2.14)