Turn this in at the start of recitation on Friday, October 31.

• Optional: Calculate the following products of permutations:
  
  – \((1, 2, 3)(2, 4, 5) = (1, 4, 5, 2, 3)\)
  
  – \((3, 4)(1, 2, 3, 4, 5, 6)(4, 3) = (1, 2, 4, 3, 5, 6)\)
  
  – \((3, 7)(1, 2, 3, 4, 5, 6)(7, 3) = (1, 2, \ldots, 7)\)
  
  – \((2, 5, 9)^{-1}(1, 2, 3, 4, 5, 6)(2, 5, 9) = ?\)

1. Show that a cycle of length \(k\) is odd if \(k\) is even and even if \(k\) is odd.

2. Is the permutation

\[(1, 9)(2, 10, 11, 6, 3)(4, 5, 12, 7) \in S_{12}\]

odd or even? Describe a general method to decide whether a product of cycles is odd or even.

3. Suppose that \(a \in S_n\) and that \(a = (a_1, \ldots, a_k)\) is a cycle of length \(k\). Show that for all \(b \in S_n\), \(b^{-1}ab\) is a cycle of length \(k\). What cycle is it?


In fact, this is part of a series of blog posts on group actions, all of which can be found by googling “gowers group actions.” I highly recommend that you take a look

**Theorem 1** (The Orbit-Stabilizer Theorem). Suppose that \(G\) is a group that acts on a set \(S\) by a homomorphism \(\rho : G \to A(S)\). For all \(a \in S\), let

\[
  \text{Stab}(a) = \{g \in G \mid \rho_g(a) = a\} \subset G
  
  \text{Orb}(a) = \{\rho_g(a) \mid g \in G\} \subset S
\]

Then,

• For all \(a \in S\), \(\text{Stab}(a)\) is a group.

• For all \(a \in S\), there is a bijective correspondence between \(\text{Orb}(a)\) and the set of right cosets of \(\text{Stab}(a)\). Namely,

\[
  s \in \text{Orb}(a) \quad \longleftrightarrow \quad \{g \in G \mid \rho_g(a) = s\}.
\]

• \(o(G) = |\text{Orb}(a)| \cdot |\text{Stab}(a)|\), where \(|\text{Orb}(a)|\) is the number of elements in \(\text{Orb}(a)\).

4. Let

\[
  S = \{(x, y) \mid -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}
\]

be the square in \(\mathbb{R}^2\).

The group \(D_8\) of symmetries of the square acts on \(S\). (For instance, if we call the four rotations \(R_0, R_{90}, R_{180}, R_{270}\), then \(\rho_{R_0} : S \to S\) is the map \(\rho_{R_0}(x, y) = (x, y), \rho_{R_{90}}(x, y) = (-y, x)\), etc.) For each of the points in the picture, draw the orbit of the point and describe its stabilizer. Verify that for each of the points, \(o(D_8) = |\text{Orb}(v)| \cdot |\text{Stab}(v)|\).
5. Previously, we saw that
\[ H = \{ f_{m,0} \mid m \in \mathbb{R}^* \} \]
is a subgroup of \( A_1 \) (the group of affine functions, not the alternating group) and \( H \) is not normal (see Assignment 3 and 4 for definitions). Show that \( H = \text{Stab}(0) \). By the orbit-stabilizer theorem, the right cosets of \( H \) are all of the form
\[ R_t = \{ f \in A_1 \mid f(0) = t \}. \]

Show that the left cosets of \( H \) are all of the form
\[ L_t = \{ f \in A_1 \mid f(t) = 0 \}. \]