Assignment 6 – Due Friday, October 17

Turn this in at the start of recitation on Friday, October 17.

1. Read Section 2.9 of Herstein.

2. Suppose that $H$ and $N$ are normal subgroups of $G$, that $H \cap N = \{ e \}$, and $HN = G$.
   (a) Show that for every $g \in G$, there are unique elements $h \in H$, $n \in N$ such that $g = hn$.
   (b) Show that $hn = nh$ for every $h \in H$ and $n \in N$. (Hint: Show that $hnh^{-1}n^{-1} \in H$ and $hnh^{-1}n^{-1} \in N$.) By the theorem in class, this implies that $G \cong H \times N$.

3. Show that $\mathbb{Z}_{12}$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_4$. Is $\mathbb{Z}_9$ isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$? Why or why not?

4. List all the homomorphisms from $\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}_6$ to $\mathbb{Z}/9\mathbb{Z} = \mathbb{Z}_9$. How are these homomorphisms related to multiplication by 0, 3, and 6? Why isn’t $f(6\mathbb{Z} + n) = 9\mathbb{Z} + 2n$ a homomorphism from $\mathbb{Z}/6$ to $\mathbb{Z}/6$, even though $g(n) = 2n$ is a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$?

5. In class, I asked the question:

   Suppose $f : G_1 \rightarrow G_2$ is a homomorphism, $N_1 \triangleleft G_1$, and $N_2 \triangleleft G_2$. When does $f$ induce a homomorphism $\bar{f} : G_1/N_1 \rightarrow G_2/N_2$?

   In this problem, we will answer that question.

   (a) Suppose $G_1$, $G_2$ are groups and that $N_1 \subset G_1$, $N_2 \subset G_2$ are normal subgroups. Let $f$ be a homomorphism. When is $f(N_1g) = N_2f(g)$ a well-defined homomorphism $G_1/N_1 \rightarrow G_2/N_2$? When $f$ is a well-defined homomorphism, we call it the homomorphism induced by $f$.

   (b) Let $G_1 = G_2 = \mathbb{R}$ and let $N_1 = N_2 = \mathbb{Z}$. For every $t \in \mathbb{R}$, the map $f_t : \mathbb{R} \rightarrow \mathbb{R}$ given by $f_t(x) = tx$ is a homomorphism. When does $f_t$ induce a homomorphism $\bar{f}_t : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$? Give a formula for $\bar{f}_t$. Graph $\bar{f}_2$ on the axes below:

   

   (c) Show that if $f$ is surjective, then $\bar{f}$ is surjective (onto). Show that if $f^{-1}(N_2) = N_1$, then $\bar{f}$ is injective (one-to-one).

   (d) Suppose that $K$ and $N$ are normal subgroups of $G$ and $K \subset N$. Use part (c) to show that the canonical homomorphism $\pi : G \rightarrow G/K$ induces an isomorphism $\bar{\pi} : G/N \rightarrow (G/K)/(N/K)$. Give an example of three groups $G$, $N$, and $K$ that satisfy the hypotheses and describe the isomorphism $\bar{\pi}$.

6. Herstein: p. 70, #6, 10