Assignment 5 – Due Friday, October 10

Turn this in at the start of recitation on Friday, October 10.

1. Read Section 2.8 of Herstein.

2. Linear algebra and group theory
   Vector spaces (with addition as their operation) and linear maps are good examples of groups and homomorphisms; in this exercise, we’ll try to view them in the context of group theory.
   (a) Suppose that $V \subset \mathbb{R}^n$ is a subspace. Show that $V$ is a normal subgroup of $\mathbb{R}^n$.
   (b) Define scalar multiplication for the quotient group $\mathbb{R}^n/V$ to make it a vector space. (Make sure your operation is well-defined, but you don’t need to check the vector space properties.)
   (c) Let $V^\perp$ be the orthogonal complement of $V$. Prove that $V^\perp$ is isomorphic to $\mathbb{R}^n/V$.
   (d) Suppose that $A \in M_{m,n}(\mathbb{R})$ is an $m \times n$ matrix and consider the homomorphism $f_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$
   
   $f_A(\bar{v}) = A\bar{v}$.
   Use the First Isomorphism Theorem to prove that $f_A(\mathbb{R}^n) \cong K^\perp$, where $K = \ker(f_A)$.
   (e) In linear algebra, you prove the following theorem:
   \textbf{Theorem 1} Let $A \in M_{m,n}(\mathbb{R})$ and $\bar{b} \in \mathbb{R}^m$. If $\bar{x}_0$ is a particular solution to the equation
   
   $A\bar{x} = \bar{b}$, \hspace{1cm} (1)
   
   and $N \in \mathbb{R}^n$ is the nullspace of $A$, then the general solution is of the form
   
   $\bar{x} = \bar{x}_0 + \bar{n}$,
   
   where $\bar{n} \in N$.
   Suppose that $G$ and $H$ are groups, that $f : G \rightarrow H$ is a homomorphism and $b \in H$. State and prove a generalization of Theorem 1 that describes the solution set of
   
   $f(x) = b$. \hspace{1cm} (2)

3. Herstein, p. 64: #3

4. Let $\mathbb{Q}$ be the additive group of rational numbers (i.e., the group whose elements are the set of rational numbers and whose operation is addition). Describe all the homomorphisms from $\mathbb{Q}$ to $\mathbb{Q}$.

5. Is there a surjective (onto) homomorphism from $\mathbb{Q}$ to the group $\mathbb{Z}_2$? Is there a subgroup of $\mathbb{Q}$ of index 2? Why or why not?

6. Suppose that $G$ and $H$ are finite groups and that $o(G)$ and $o(H)$ are relatively prime (i.e., they have no common divisors except 1). Show that the only homomorphism from $G$ to $H$ is the one that sends every element of $G$ to the identity element.