1. Assume first that \( f \) is injective. Let \( A, B \subset X \). If \( y \in f(A \cap B) \) then there exists \( x \in A \cap B \) with \( f(x) = y \). Since \( x \in A \) and \( x \in B \) it follows that \( y \in f(A) \cap f(B) \). If \( y \in f(A) \cap f(B) \) then there exist \( x \in A, x' \in B \) with \( y = f(x) = f(x') \). Since \( f \) is injective, we must have \( x = x' \), thus \( x \in A \cap B \) and \( y \in f(A \cap B) \). This proves \( f(A \cap B) = f(A) \cap f(B) \).

If \( f \) is not injective, then there are \( a, b \in X \), \( a \neq b \) with \( f(a) = f(b) \). Chose \( A = \{a\} \), \( B = \{b\} \). Then \( A \cap B = \emptyset \), thus \( f(A \cap B) = \emptyset \). On the other hand, \( f(A) \cap f(B) = \{f(a)\} \neq \emptyset \).

2. Observe that \( \varphi : P(\mathbb{Z}) \to \mathcal{F} \) defined by \( \varphi(A)(x) = 1 \) if \( x \in A \) and \( \varphi(A)(x) = 0 \) if \( x \notin A \) is a bijection. Thus \( \mathcal{F} \) is uncountable.

3. Since \( f \) is monotone, the left and right limits \( \lim_{t \to x^\pm} f(t) \) exist for each \( x \in \mathbb{R} \). Let \( j(x) := \lim_{t \to x^+} f(t) - \lim_{t \to x^-} f(t) \). Note that \( x \) is a point of discontinuity of \( f \) if and only if \( j(x) > 0 \). Thus, the set of discontinuities can be written as
\[
\bigcup_{n=1}^\infty \bigcup_{m=1}^\infty \{x \in [-m,m] | j(x) > 1/n\},
\]
which is a countable union of countable sets. This proves the claim.

4. Since \( Y \subset X \) we see that every number \( a \in \mathbb{R} \) that is an upper bound for \( X \) also is an upper bound for \( Y \). In particular, \( \sup X \) is an upper bound for \( Y \). Thus \( \sup Y \leq \sup X \).

5. We compute
\[
\lim_{n \to \infty} \frac{8n^5 + 42n^3 + 117}{2n^5 + n^2 + 1} = \lim_{n \to \infty} \frac{4 + 21n^{-2} + \frac{117}{2}n^{-5}}{1 + \frac{1}{2}n^{-3} + \frac{1}{2}n^{-5}} = 4 + 21 \lim_{n \to \infty} n^{-2} + \frac{117}{2} \lim_{n \to \infty} n^{-5} = 4.
\]

6. Let \( M < \infty \). Since \( \{a_n\} \) is unbounded there exists an \( N \) such that \( a_N > M \). Since \( \{a_n\} \) is also increasing, it follows that \( a_n > M \) for all \( n \geq N \). This proves \( \lim_{n \to \infty} a_n = \infty \).

7. One should check first that \( \{b_n\}_{n=5}^\infty \) is increasing and bounded, thus convergent. The limit can be computed as \( \lim b_n = \lim a_{n+1} \left( \lim a_n \right)^{-1} = 1 \).

8. By definition, convergence of the series \( \sum_{j=1}^\infty a_j \) means that the sequence of partial sums \( s_k = \sum_{j=1}^k a_j \) converges. Thus, given any positive integer \( n \), the sequence \( s_k - s_{n-1} \) also converges, i.e. the sequence \( \sum_{j=n}^\infty a_j \) converges. Let \( s := \lim_{k \to \infty} s_k \). Note that \( s = s_{n-1} + \sum_{j=n}^\infty a_j \). We conclude that \( \lim_{n \to \infty} \sum_{j=n}^\infty a_j = 0 \).

9. The series \( \sum_{n=1}^\infty \frac{n+1}{n^3+10n} \) converges, since \( \sum_{n=1}^\infty \frac{100}{n^3} < \infty \) and \( \frac{n+1}{n^3+10n} \leq \frac{100}{n^3} \) for all \( n \).

10. Note that \( n^{1/n} < 2 \) for all \( n \). Thus \( \frac{1}{n^{1+1/n}} > \frac{1}{2n} \), and \( \sum_{n=1}^\infty \frac{(-1)^n}{n^{1+1/n}} \) does not converge absolutely. By the criterion for alternating series, it converges conditionally.

11. The statement \( \lim_{x \to \infty} f(x) = L \) means that for every \( \varepsilon > 0 \) there exists an \( M < \infty \) such that if \( x > M \) then \( |f(x) - L| < \varepsilon \). Substituting \( y = 1/x \) the latter can be rewritten as: If \( 0 < y < 1/M \) then \( |f(1/y) - L| < \varepsilon \). Thus, the statement is equivalent to \( \lim_{y \to 0^+} f(1/y) = L \).
12. Since $f$ is a rational function with nowhere vanishing denominator, it is continuous. Since the interval $[21, 42]$ is compact, it follows that $f$ has a maximum, i.e. there exists an $x_0 \in [21, 42]$ such that $f(x) \leq f(x_0)$ for all $x \in [21, 42]$.

13. For example $f(x) = 1/x, x \in (0, 1]$ is continuous, but not uniformly continuous. Assume now $f$ is a continuous function on a compact interval $I = [a, b]$ and let $\varepsilon > 0$. By continuity, for every $x \in I$ there exists a $\delta_x > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ for all $y \in I$ with $|x - y| < \delta_x$. Let $I_x = (x - \delta_x/2, x + \delta_x/2)$. By compactness, the open cover $\bigcup_{x \in I} I_x \supset I$ has a finite subcover, i.e. there exist $x_1, \ldots, x_k \in I$ such that $I_{x_1} \cup \ldots \cup I_{x_k} \supset I$. Let $\delta := \min\{\delta_{x_1}, \ldots, \delta_{x_k}\}$. Then for any $x, y \in I$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

14. Let $I$ be a compact interval. Suppose towards a contradiction that $\{x_n\}$ is a sequence in $I$ without convergent subsequence. Then for every $x \in I$ there is an $\varepsilon_x > 0$ such that $I_x := (x - \varepsilon_x, x + \varepsilon_x)$ contains only finitely many $x_n$. It follows that $\bigcup_{x \in I} I_x \supset I$ is an open cover, without finite subcover; this is a contradiction.