Intro to Math Analysis: HW2

Due: Sep 25 (Thursday)

1. Doubling the cube
Prove there is no rational number \( r \) such that \( r^3 = 2 \).

2. Fields
A field is a set \( F \) satisfying the axioms A1-A11 from the textbook.
   
i. Give at least four examples of a field.
   
ii. Prove that for every field, additive and multiplicative inverse elements are unique.

3. Least upper bound
Let \( X \) and \( Y \) be sets of real numbers with least upper bound \( a \) and \( b \), respectively. Prove that \( a + b \) is the least upper bound for the set

\[
X + Y = \{x + y | x \in X, y \in Y\}.
\]

4. Greatest lower bound
Prove that every nonempty subset \( X \subset \mathbb{R} \) which is bounded below has a unique greatest lower bound.

5. Limit of absolute values
Given a sequence of real numbers with \( \lim_{n \to \infty} a_n = L \), prove that \( \lim_{n \to \infty} |a_n| = |L| \).

6. Number of subsequential limits
Give an example of a sequence which has "many" different subsequential limits. Is the number of subsequential limits always countable?