In this final lecture, we discuss some aspects of the convergence of Riemannian manifolds. For ease of presentation, we assume uniform diameter bounds.\footnote{The natural notion of convergence without diameter bounds is pointed convergence. Roughly speaking, this is uniform convergence on compact subsets around the basepoint.}

1 Gromov-Hausdorff convergence

We first discuss Gromov-Hausdorff convergence. This is a notion of convergence for sequences of metric spaces, and can be thought of as convergence in a weak sense.

**Definition 1.1 (Gromov-Hausdorff convergence)**

A sequence \((X_i, d_i)\) of compact metric spaces Gromov-Hausdorff converges to a compact metric space \((X, d)\), if there exists a sequence of \(\varepsilon_i\)-approximations \(\phi_i : X_i \to X\), see Definition 1.2, with \(\varepsilon_i \to 0\).

**Definition 1.2 (\(\varepsilon\)-approximation)**

Let \(X, Y\) be two compact metric spaces. A map \(\phi : X \to Y\) is called an \(\varepsilon\)-approximation if (i) \(|d(x, x') - d(\phi(x), \phi(x'))| < \varepsilon\) for all \(x, x' \in X\), and (ii) for all \(y \in Y\) there exists an \(x \in X\) with \(d(y, \phi(x)) < \varepsilon\).

**Example 1.3 (Collapsing tori)**

Consider the spheres \(S^k\) and \(S^{n-k}\) with the standard metric. Then, the sequence \(S^k \times S^{n-k}\) Gromov-Hausdorff converges to \(S^k\).

**Example 1.4 (Blowdown of Cayley graph)**

Let \(M\) be the Cayley graph of \(\mathbb{Z}^n\) equipped with the word metric. Then \(\frac{1}{n} M\) converges in the pointed Gromov-Hausdorff sense to \(\mathbb{R}^n\) with the \(\ell^1\)-metric.

**Remark 1.5 (Groups of polynomial growth)**

Gromov proved that groups of polynomial growth are virtually nilpotent. His key and highly creative insight was to introduce a notion of convergence for metric spaces and to apply it in group theory; see also the more elementary proof by Kleiner.
Let us now discuss Gromov’s precompactness theorem.

**Theorem 1.6** (Gromov’s precompactness theorem)

For every sequence $(M_i, g_i)$ of Riemannian manifolds with uniform lower Ricci and upper diameter bound, i.e. $\text{Rc}(g_i) \geq -\kappa$ and $\text{diam}(M_i, g_i) \leq D$, there exists a subsequence that Gromov-Hausdorff converges to a compact metric space $(X, d)$.

The key is Bishop-Gromov volume comparison. We write $V_\kappa(r)$ for the volume of a ball of radius $r$ in the model space with constant sectional curvature $\kappa$.

**Theorem 1.7** (Bishop-Gromov volume comparison)

If $(M, g)$ is a Riemannian manifold with $\text{Rc}(g) \geq (n - 1)\kappa$, $p \in M$, and $r_1 \leq r_2$ then

$$\frac{\text{Vol} B_p(r_2)}{V_\kappa(r_2)} \leq \frac{\text{Vol} B_p(r_1)}{V_\kappa(r_1)}.$$  \hspace{1cm} (1.1)

**Proof of Theorem 1.6.** For each $k$, let $Q^k_i$ be a maximal set of points in $M_i$ with distance from each other at least $1/k$. We choose the sets such that they are monotone in $k$. By Bishop-Gromov volume comparison (Theorem 1.7), the number $N^k_i$ of points in $Q^k_i$ is bounded by

$$N^k_i \leq \frac{\text{Vol}(M_i)}{\min_{p \in Q^k_i} \text{Vol}(B_p(1/2k))} \leq \frac{V_\kappa(D)}{V_\kappa(1/2k)}.$$ \hspace{1cm} (1.2)

For each $k$, after passing to a subsequence we can assume that $N^k_i = N^k$ is constant. Let us label the elements of $Q^k_i$ by $p_1, \ldots, p_{N^k}$. After passing to another subsequence, the numbers $d_i(p_a, p_b)$ converge to some number $d(a, b)$ for all $a, b = 1, \ldots, N^k$. Thus, passing to a diagonal subsequence we get Gromov-Hausdorff convergence to the completion of $(\mathbb{N}, d)$.

**Remark 1.8** (Weakening the assumptions)

The proof shows that instead of $\text{Rc}(g_i) \geq -\kappa$, it is enough to assume that there exists a uniform bound $N(\varepsilon) < \infty$ ($\varepsilon > 0$) for the number of disjoint $\varepsilon$-balls in $(M_i, d_i)$.

**Proof of Theorem 1.7.** Let $r(x) = d(x, p)$ be the distance from a fixed point $p \in M$. Away from the cut locus, since $|\nabla r| = 1$, the Bochner-formula implies

$$\text{Hess} r^2 + \partial_r \Delta r + \text{Rc}(\partial_r, \partial_r) = 0.$$ \hspace{1cm} (1.3)

Using Cauchy-Schwarz and the assumption $\text{Rc} \geq (n - 1)\kappa$ this implies

$$\frac{1}{n-1} \Delta r + \partial_r \Delta r + (n - 1)\kappa \leq 0.$$ \hspace{1cm} (1.4)

In terms of $u = \frac{1}{n-1} \Delta r$, this inequality takes the form $u' \geq 1 + \kappa u^2$. Also note that
$\triangle r \to \frac{n-1}{r}$ when $r \to 0$; thus $u \to r$. By integration we obtain

$$\triangle r \leq \triangle^{n}r = \left\{ \begin{array}{ll}
(n-1)\sqrt{\kappa} \cot \sqrt{\kappa r} & \text{for } \kappa > 0 \\
(n-1)/r & \text{for } \kappa = 0 \\
(n-1)\sqrt{-\kappa} \coth \sqrt{-\kappa r} & \text{for } \kappa < 0.
\end{array} \right. \quad (1.5)$$

Speaking more geometrically, we have proved the inequality

$$H(r, \theta) \leq H^{n}(r), \quad (1.6)$$

where $H$ denotes the mean curvature of the hypersurface at distance $r$ from $p$. In geodesic polar coordinates, the volume form can be written as

$$dVol = A(r, \theta)dr \wedge d\theta, \quad (1.7)$$

where $d\theta$ is the standard volume form on $S^{n-1}$. Finally, observing that

$$H(r, \theta) = \frac{A'(r, \theta)}{A(r, \theta)}, \quad (1.8)$$

the Bishop-Gromov inequality follows by integration (the cut locus only helps).

**Example 1.9** (Blowdown of Eguchi-Hanson metric)

Consider $TS^2$ with the Eguchi-Hanson metric $g$. This can be described as follows. Recall that $TS^2 \setminus S^2$ is diffeomorphic to $\mathbb{R}_+ \times SO_3 = \mathbb{R}_+ \times (S^3/\mathbb{Z}_2)$. Let $\lambda, \mu, \nu$ be a left-invariant orthonormal coframe on $S^3/\mathbb{Z}_2$. Then

$$g = \frac{dr^2}{1 - r^{-4}} + r^2(\lambda^2 + \mu^2 + (1 - r^{-4})\nu^2), \quad (1.9)$$

for $r > 1$, and in fact this closes up smoothly for $r \to 1$ to a metric on $TS^2$. A direct computation shows that the Eguchi-Hanson metric is Ricci-flat, i.e.

$$\text{Rc}(g) = 0. \quad (1.10)$$

Moreover, $(TS^2, g)$ is asymptotic to flat $\mathbb{R}^4/\mathbb{Z}_2$. Thus, $(M_i, g_i) = (TS^2, \frac{1}{i}g)$ is a sequence of Ricci-flat manifolds that converges in the pointed Gromov-Hausdorff sense to $\mathbb{R}^4/\mathbb{Z}_2$. The limit is a metric cone, with a singularity at the tip.

**Example 1.10** (Singular set of codimension four)

The sequence of $n$-dimensional Ricci-flat manifolds $(TS^2, \frac{1}{i}g) \times \mathbb{R}^{n-4}$ converges to $\mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}^{n-4}$. The limit is noncollapsed, and its singular set has dimension $n-4$.

**Remark 1.11** (Cheeger-Colding theory)

There is a deep structure theory for limit spaces $(X, d, m)$ with Ricci bounded below,\footnote{Indeed, since $N = \text{grad } r$, we have $\triangle r = \text{tr} \text{Hess } r = \sum_{i=1}^{n-1} \langle \nabla e_i N, e_i \rangle = H(r, \theta)$.} due to Cheeger-Colding and others, c.f. the lecture by Cheeger.\footnote{$(X, d)$ comes naturally equipped with a measure $m$ that arises as limit of volume measures.}
Remark 1.12 (Ricci bounded below via optimal transport)

There is also a recent synthetic treatment of Ricci bounded below for metric measure spaces \((X,d,m)\), due to Lott-Villani, Sturm and Ambrosio-Gigli-Savare.

2 Cheeger-Gromov convergence

Finally, let us discuss Cheeger-Gromov convergence. This is a notion of convergence for Riemannian manifolds, much stronger than Gromov-Hausdorff convergence.

Definition 2.1 (Cheeger-Gromov convergence)

A sequence \((M_i,g_i)\) of closed Riemannian manifolds \(\text{Cheeger-Gromov converges}\) to a closed Riemannian manifold \((M,g)\) (with regularity \(C^{k,\alpha}\)), if there exists a sequence of diffeomorphism \(\phi_i : M \to M_i\), such that \(\phi_i^* g_i \to g\) in \(C^{k,\alpha}\).

The fundamental result is the Cheeger-Gromov compactness theorem.

Theorem 2.2 (Cheeger-Gromov compactness theorem)

If \((M_i,g_i)\) is a sequence of closed Riemannian manifolds with uniform bounds

\[
|K_{g_i}| \leq \kappa, \quad \text{Vol}(M,g_i) \geq v, \quad \text{diam}(M,g_i) \leq D, \tag{2.1}
\]

then there exists a subsequence \((M_{i_k},g_{i_k})\) that Cheeger-Gromov converges to a closed Riemannian manifold \((M,g)\) (with regularity \(C^{1,\alpha}\)).

An immediate consequence is Cheeger’s bound for the number of diffeomorphism classes of closed manifolds that admit a Riemannian metric with definite bounds on curvature (two-sided), diameter (from above) and volume (from below).

Corollary 2.3 (Cheeger’s diffeomorphism finiteness)

There exists a function \(N = N(\kappa,D,v) < \infty (\kappa < \infty, D < \infty, v > 0)\) such that the number of diffeomorphism classes of closed manifolds \(M\), that admit a Riemannian metric \(g\) with \(|K_g| \leq \kappa, \text{diam}(M,g) \leq D\) and \(\text{Vol}(M,g) \geq v\), is at most \(N\).

The strategy to prove the Cheeger-Gromov compactness theorem is as follows. One argues that the assumptions (2.1) imply that for some \(r = r(\kappa,v,D) > 0\) we can cover each \((M_i,g_i)\) with at most \(C = C(\kappa,v,D) < \infty\) charts \(x_i : B_r \subset \mathbb{R}^n \to x_i(M) \subset M\), such that the metrics satisfy the uniform bounds

\[
\frac{1}{2} \delta_{ab} \leq (x_i^* g_i)_{ab} \leq 2 \delta_{ab}, \quad \|x_i^* g_i\|_{C^{1,\beta}} \leq C \tag{2.2}
\]

and the transition functions satisfy the uniform estimate

\[
\|x_i \circ x_j^{-1}\|_{C^{2,\beta}} \leq C, \tag{2.3}
\]

for some \(\beta > \alpha\). The theorem then follows by passing to a subsequential limit of the metrics in coordinates and their transition functions.