Homework 1
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1. (a) Show that for any vectors $|u_1\rangle, |u_2\rangle, |v_1\rangle, |v_2\rangle$,

   $$(\langle u_1| \otimes \langle u_2|)(|v_1\rangle \otimes |v_2\rangle) = \langle u_1|v_1\rangle \cdot \langle u_2|v_2\rangle.$$

   Conclude that $\||u_1\rangle \otimes |u_2\rangle\| = \||u_1\rangle\| \cdot \||u_2\rangle\|$ and so if $|u_1\rangle, |u_2\rangle$ are unit vectors, so is $|u_1\rangle|u_2\rangle = |u_1\rangle \otimes |u_2\rangle$.

   (b) Calculate $\mathsf{X} \otimes \mathsf{Z}, \mathsf{Z} \otimes \mathsf{X}, \mathsf{H} \otimes \mathsf{H}$. Is the tensor product commutative?

   (c) Show that if U_1, U_2 are unitary matrices then $U_1 \otimes U_2$ is also unitary. Find its inverse.

   (d) Show that $(U_1 \otimes U_2)(|u\rangle \otimes |v\rangle) = (U_1|u\rangle) \otimes (U_2|v\rangle)$ and that $U_1 \otimes U_2 = (U_1 \otimes I)(I \otimes U_2) = (I \otimes U_2)(U_1 \otimes I)$.

2. (a) Assume we measure the first qubit of an EPR pair in the $|\pm\rangle$ basis. For each outcome, what is the state of the second qubit? Generalize this to a measurement in an arbitrary basis on the first qubit.

   (b) Let U be an arbitrary 1-qubit unitary and let $|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$ (this is known as the *singlet* state). What state is obtained by applying U to each of the qubits of $|\psi\rangle$? In other words, compute $(U \otimes U)|\psi\rangle$.

3. (a) Alice has the 2-qubit state $\alpha|00\rangle + \beta|11\rangle$. Show how she can teleport it to Bob using 2 classical bits of communication and one shared EPR pair.

   (b) Alice has an arbitrary 2-qubit state $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$. Show how she can teleport it to Bob using 4 classical bits of communication and two shared EPR pairs. What is the generalization to n qubits?

4. (a) Show that $U^2 = I$ for $U \in \{\mathsf{X}, \mathsf{Y}, \mathsf{Z}, \mathsf{H}, \mathsf{CNOT}\}$. Give a characterization of all 2-dimensional unitaries satisfying $U^2 = I$ in terms of their eigenvalues.

   (b) Compute HXH.

   (c) Let CZ be the 2-qubit gate defined by $\mathsf{CZ}|00\rangle = |00\rangle, \mathsf{CZ}|01\rangle = |01\rangle, \mathsf{CZ}|10\rangle = |10\rangle, \mathsf{CZ}|11\rangle = -|11\rangle$ (controlled-Z gate). Show how to implement CZ using CNOT and H.

   (d) Let U be a 2-dimensional unitary for which $U^2 = I$. Try to generalize your solution to (4c) by showing how to implement the controlled-U gate using CNOT and arbitrary 1-qubit gates. The controlled-U gate is given by the matrix

   $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & U_{10} & U_{11} \end{pmatrix}$$

5. Let U be an arbitrary 1-qubit unitary. Show that there exists a V such that $V^2 = U$. Show that for such a V, the circuit equivalence in Figure 1 holds. On the left hand side we have the double controlled U, i.e., U is applied when $|j_0 j_1\rangle = |11\rangle$.

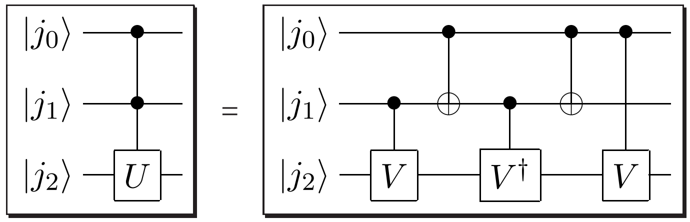

Figure 1: Doubly-controlled U

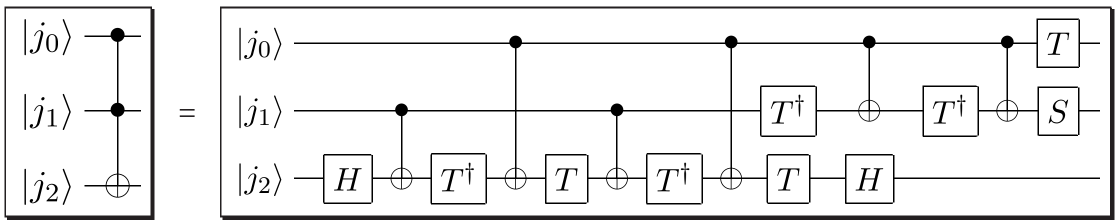

Figure 2: Toffoli

6. (a) Show that the circuit equivalence in Figure 2 holds. On the left hand side, we have the Toffoli gate, i.e., the doubly-controlled X gate. On the right hand side, S (known as the *phase gate*) and T (known as the $\pi/8$-gate) are given by

$$\mathsf{S} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad \mathsf{T} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

   (b) Show how to implement an n-controlled X using $2n-3$ Toffoli gates. You can assume you have ancilla qubits initialized to $|0\rangle$. Note that you must return the ancilla to their original state (why?).