Lecture 9
Fourier Transform

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In this lecture we describe some basic facts of Fourier analysis that will be needed later. The first section discusses the Fourier transform, and the second discusses the Fourier series. We start each section with the more familiar case of one-dimensional functions and then extend it to the higher dimensional case. As a general rule, we will not worry too much about issues of convergence, differentiability etc., as these will always be satisfied in our applications.

1 Fourier Transform

1.1 The one-dimensional case

DEFINITION 1 *We define $L^1(\mathbb{R})$ as the set of functions $f : \mathbb{R} \to \mathbb{C}$ satisfying $\int_{-\infty}^{\infty} |f(x)| dx < \infty$.*

DEFINITION 2 *For a function $f \in L^1(\mathbb{R})$ define its Fourier transform as the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ given by*

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y} dx$$

For example, the Fourier transform at point 0 is $\hat{f}(0) = \int_{-\infty}^{\infty} f(x) dx$.

EXAMPLE 1 Define

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < a \\ 0 & \text{otherwise} \end{cases}.$$

Then,

$$\hat{f}(y) = \int_{-a}^{a} e^{-2\pi i x y} dx = \frac{e^{-2\pi i a y} - e^{2\pi i a y}}{-2\pi i y} = \frac{\sin(2\pi a y)}{y\pi}.$$

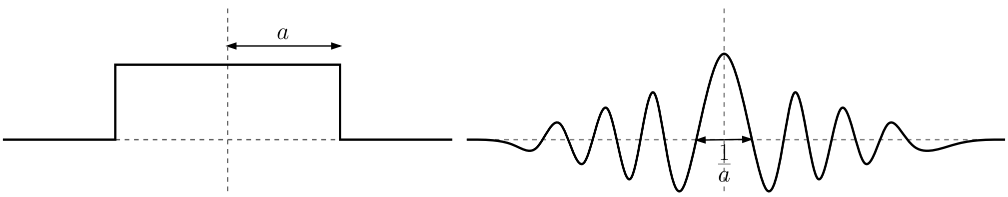

Figure 1: $f(x)$ and $\hat{f}(y)$

EXAMPLE 2 Let $f(x) = e^{-\pi(\frac{x}{s})^2}$ for some $s > 0$. Then,

$$\begin{aligned} \hat{f}(y) &= \int_{-\infty}^{\infty} e^{-\pi(\frac{x}{s})^2} e^{-2\pi i x y} dx = \int_{-\infty}^{\infty} e^{-\pi(\frac{x^2}{s^2} + 2ixy)} dx \\ &= \int_{-\infty}^{\infty} e^{-\pi(\frac{x}{s} + iys)^2} e^{-\pi(ys)^2} dx = e^{-\pi(ys)^2} \int_{-\infty}^{\infty} e^{-\pi(\frac{x}{s} + iys)^2} dx. \end{aligned}$$

We now perform a (complex) change of variable $z = \frac{x}{s} + iys$ (which is possible by Cauchy's theorem), and see that the above is equal to

$$s \cdot e^{-\pi(ys)^2} \int_{-\infty}^{\infty} e^{-\pi z^2} dz = s \cdot e^{-\pi(ys)^2}.$$

Notice that for $s = 1$ we get that $f(x) = e^{-\pi x^2}$ satisfies $\hat{f} = f$.

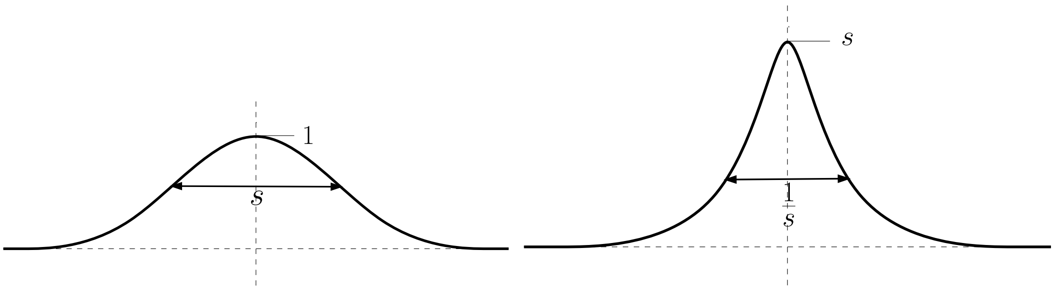

Figure 2: $f(x)$ and $\hat{f}(y)$

The following theorem lists some of the most important properties of the Fourier transform. The first property shows that the Fourier transform is linear. The third and fourth properties show that under the Fourier transform, translation becomes multiplication by phase and vice versa. The sixth property shows that scaling a function by some $\lambda > 0$ scales its Fourier transform by $1/\lambda$ (together with the appropriate normalization). The seventh property shows that under the Fourier transform, convolution becomes multiplications and vice versa, where we define the convolution of two functions as $f * g(y) = \int_{\mathbb{R}} f(x)g(y-x)dx$. The last property shows that the Fourier transform of the derivative of a function can be obtained by simply multiplying the Fourier transform of the function by $2\pi iy$.

THEOREM 1 *For all $f, g \in L^1(\mathbb{R})$, $x, y, z \in \mathbb{R}$, the following holds:*

1. $\widehat{f+g} = \hat{f} + \hat{g}$ *and for all* $\alpha \in \mathbb{C}$, $\widehat{(\alpha f)} = \alpha \hat{f}$
2. *if* $\bar{f}$ *is the complex conjugate of* f *then* $(\widehat{\bar{f}})(y) = \overline{\hat{f}(-y)}$
3. *if* $h(x) := f(x+z)$ *then* $\hat{h}(y) = \hat{f}(y) \cdot e^{2\pi izy}$
4. *if* $h(x) := e^{2\pi izx} f(x)$ *then* $\hat{h}(y) = \hat{f}(y-z)$
5. $|\hat{f}(y)| \leq \int_{-\infty}^{\infty} |f(x)|dx$
6. $\forall \lambda > 0$, *define* $h(x) := \lambda f(\lambda x)$ *then* $\hat{h}(y) = \hat{f}(\frac{y}{\lambda})$
7. $\widehat{f * g} = \hat{f} \cdot \hat{g}$ *and* $\widehat{f \cdot g} = \hat{f} * \hat{g}$
8. *if* $h(x) = f'(x) \in L^1(\mathbb{R})$ *then* $\hat{h}(y) = 2\pi iy\hat{f}(y)$

PROOF: Most items are easy to verify. We only include a proof of two of them.

3.
$$\hat{h}(y) = \int_{-\infty}^{\infty} f(x+z) \cdot e^{-2\pi ixy} dx = \int_{-\infty}^{\infty} f(x) e^{-2\pi i(x-z)y} dx = e^{2\pi izy} \hat{f}(y)$$

8. Since $f' \in L^1(\mathbb{R})$, $\lim_{x\to\infty} f(x) = f(0) + \int_0^\infty f'(x)dx$ exists, and since $f \in L^1(\mathbb{R})$ this limit must be zero. Similarly $\lim_{x\to-\infty} f(x) = 0$. Hence, using integration by parts
$$\int_{-\infty}^{\infty} f'(x) e^{-2\pi ixy} dx = -\int_{-\infty}^{\infty} (-2\pi iy) f(x) e^{-2\pi ixy} dx = 2\pi iy\hat{f}(y)$$

□

The following theorem, known as the inversion formula, shows that a function can be recovered from its Fourier transform. The proof is omitted.

THEOREM 2 *If both $f, \hat{f} \in L^1(\mathbb{R})$ and f is continuous then $f(x) = \int_{-\infty}^{\infty} \hat{f}(y) e^{2\pi i x y} dy$*

1.2 The n-dimensional case

We now extend the Fourier transform to functions on $\mathbb{R}^n$. Define $L^1(\mathbb{R}^n)$ as the set of functions $f : \mathbb{R}^n \to \mathbb{C}$ satisfying $\int_{\mathbb{R}^n} |f(x)| dx < \infty$. We also define, for $x, y \in \mathbb{R}^n$, $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ and $\|x\| = \sqrt{\langle x, x \rangle}$. We now define the n-dimensional Fourier transform.

DEFINITION 3 *For $f \in L^1(\mathbb{R}^n)$ define $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by*

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, y \rangle} dx$$

As shown below, all properties listed in Theorem 1 can be extended to the n-dimensional case. The proof is essentially the same as that in the one-dimensional case. The only new property is the last one: it says that if an n-dimensional function can be factored as the product of n one-dimensional functions, then its Fourier transform is the product of the individual Fourier transforms. The proof of this is left to the reader.

THEOREM 3 *For all $f, g \in L^1(\mathbb{R}^n)$, $x, y, z \in \mathbb{R}^n$, the following holds:*

1. $\widehat{f+g} = \hat{f} + \hat{g}$ *and for all* $\alpha \in \mathbb{C}$, $\widehat{(\alpha f)} = \alpha \hat{f}$
2. *if $\bar{f}$ is the complex conjugate of f then* $(\widehat{\bar{f}})(y) = \overline{\hat{f}(-y)}$
3. *if* $h(x) := f(x+z)$, $\hat{h}(y) = e^{2\pi i \langle y, z \rangle} \hat{f}(y)$
4. *if* $h(x) := e^{2\pi i \langle x, z \rangle} f(x)$ *then* $\hat{h}(y) = \hat{f}(y - z)$
5. $|\hat{f}(y)| \leq \int_{\mathbb{R}^n} |f(x)| dx$
6. *for* $\lambda > 0$, $h(x) := \lambda^n f(\lambda x)$ *then* $\hat{h}(y) = \hat{f}(\frac{y}{\lambda})$
7. $\widehat{f * g} = \hat{f} \cdot \hat{g}$ *and* $\widehat{f \cdot g} = \hat{f} * \hat{g}$
8. *if* $\frac{\partial f}{\partial x_j}$ *exists then* $\left(\widehat{\frac{\partial f}{\partial x_j}}\right)(y) = 2\pi i y_j \hat{f}(y)$
9. *if* $f(x) = f_1(x_1) \cdots f_n(x_n)$ *then* $\hat{f}(y) = \hat{f}_1(y_1) \cdots \hat{f}_n(y_n)$

The following example will be used in future lectures.

EXAMPLE 3 Consider $\rho(x) := e^{-\pi \|x\|^2}$. Then

$$\rho(x) = e^{-\pi(x_1^2 + \cdots + x_n^2)} = e^{-\pi x_1^2} \cdots e^{-\pi x_n^2}$$

hence we obtain that $\hat{\rho}(y) = \rho(y)$. More generally, for $\rho_s(x) := e^{-\pi \|\frac{x}{s}\|^2}$, $\hat{\rho}_s(y) = s^n \rho_{\frac{1}{s}}(y)$.

We also have the following extension of the inversion formula (we omit the exact smoothness conditions required from f).

THEOREM 4 *For $f, \hat{f} \in L^1(\mathbb{R}^n)$, $f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{2\pi i \langle x, y \rangle} dy$*

2 The Fourier Series

2.1 The one-dimensional case

In this subsection, we consider functions on $\mathbb{R}$ with period 1, i.e., functions f that satisfy that $f(x+y) = f(x)$ for any $x \in \mathbb{R}, y \in \mathbb{Z}$.

DEFINITION 4 *For a function $f : \mathbb{R} \to \mathbb{C}$ with period 1 we define*[1] *its Fourier series as the function $\hat{f} : \mathbb{Z} \to \mathbb{C}$ given by*

$$\hat{f}(k) = \int_0^1 f(x)e^{-2\pi ikx}dx.$$

The value $\hat{f}(k)$ is sometimes called the kth Fourier coefficient.

Notice that unlike the Fourier transform, the Fourier series is only defined on a discrete set of points, namely $\mathbb{Z}$. The intuitive reason for this is that in a 1-periodic function, only integer frequencies appear. Moreover, it is interesting to note that the Fourier coefficients can be seen as the limit of the Fourier transform in the following sense. Consider a periodic function on $\mathbb{R}$ and restrict it to $[-R, R]$. Then, roughly speaking, as R goes to ∞, its Fourier transform converges to 0 on non-integer points and to the Fourier coefficients on integer points.

Most of the properties of the Fourier transform given in Theorem 1 also hold for the Fourier series. We mention some below.

THEOREM 5 *For any two functions f, g with period 1 we have*

1. $\widehat{f+g} = \hat{f} + \hat{g}$ *and for any* $\alpha \in \mathbb{C}$, $\widehat{\alpha f} = \alpha \hat{f}$
2. *if* $h(x) := f(x+r)$ *for some* $r \in \mathbb{R}$, *then* $\hat{h}(k) = \hat{f}(k) \cdot e^{2\pi ikr}$
3. *if* $h(x) := e^{2\pi ijx} f(x)$ *for some* $j \in \mathbb{Z}$ *then* $\hat{h}(k) = \hat{f}(k-j)$

The following is the inversion formula for the Fourier series (also known as the Fourier convergence theorem). Notice that in the case that f is continuous, the right hand side is simply $f(x)$.

THEOREM 6 *For any piecewise smooth f with period 1 we have*

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikx} = \frac{1}{2}\left(f(x+) + f(x-)\right).$$

The following theorem is known as the *Poisson summation formula*. Its proof is based on a connection between the Fourier transform and the Fourier series.

THEOREM 7 (PSF$_1$) *For a nice enough $f \in L^1(\mathbb{R})$,*

$$\sum_{j=-\infty}^{\infty} f(j) = \sum_{j=-\infty}^{\infty} \hat{f}(j).$$

Equivalently, $f(\mathbb{Z}) = \hat{f}(\mathbb{Z})$.

[1] To be precise, we should assume that $\int_0^1 |f(x)|dx$ exists. From now on, we ignore such issues of convergence.

PROOF: Given a function f, define $\varphi(t) = \sum_{j=-\infty}^{\infty} f(t+j)$. Notice that φ has period 1, and we can therefore consider its Fourier series,

$$\begin{aligned}\hat{\varphi}(k) = \int_0^1 \varphi(t)e^{-2\pi ikt}dt &= \sum_{j=-\infty}^{\infty} \int_0^1 f(t+j)e^{-2\pi ikt}dt \\ &= \sum_{j=-\infty}^{\infty} \int_0^1 f(t+j)e^{-2\pi ik(t+j)}dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-2\pi ikt}dt = \hat{f}(k).\end{aligned}$$

So, we see that $\hat{\varphi}$ is the restriction of $\hat{f}$ to the integers. Using the inversion formula we have

$$\varphi(0) = \sum_{j=-\infty}^{\infty} \hat{\varphi}(j),$$

and the theorem follows. $\square$

In the rest of this subsection, we extend our definition of the Fourier series to functions whose period is not necessarily 1. It should be noted that this extension is not strictly necessary in the sense that any function f with period λ can be transformed into a function with period 1 by simply defining $g(x) := f(\lambda x)$. Nevertheless, it serves as a good introduction to Fourier series on lattices since what we are doing here is essentially defining the Fourier series of functions that are periodic on an arbitrary one-dimensional lattice $\lambda\mathbb{Z}$ (whereas so far we only dealt with the lattice $\mathbb{Z}$).

For a function $f : \mathbb{R} \to \mathbb{C}$ with some period $\lambda > 0$ we define its Fourier series as $\hat{f} : \frac{1}{\lambda}\mathbb{Z} \to \mathbb{C}$ by

$$\hat{f}(y) = \frac{1}{\lambda}\int_0^\lambda f(x)e^{-2\pi ixy}dx.$$

The inversion formula becomes the following.

THEOREM 8

$$f(x) = \sum_{y \in \frac{1}{\lambda}\mathbb{Z}} \hat{f}(y)e^{2\pi ixy}$$

We now obtain the following extension of the Poisson summation formula (we remark that this extension can also be derived directly from Theorem 7).

LEMMA 9 (PSF$_2$) *For any $\lambda > 0$ and any nice enough function f,*

$$\sum_{x \in \lambda\mathbb{Z}} f(x) = \frac{1}{\lambda} \sum_{y \in \mathbb{Z}/\lambda} \hat{f}(y).$$

Equivalently, $f(\lambda\mathbb{Z}) = \frac{1}{\lambda}\hat{f}(\frac{1}{\lambda}\mathbb{Z})$.

PROOF: Define $\varphi(x) = \sum_{j=-\infty}^{\infty} f(x+\lambda j)$. Then φ has period λ and for $y \in \frac{1}{\lambda}\mathbb{Z}$,

$$\begin{aligned}\hat{\varphi}(y) &= \frac{1}{\lambda}\int_0^\lambda \varphi(x)e^{-2\pi ixy}dx \\ &= \frac{1}{\lambda}\sum_{j=-\infty}^{\infty}\int_0^\lambda f(x+\lambda j)e^{-2\pi ixy}dx \\ &= \frac{1}{\lambda}\int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx \\ &= \frac{1}{\lambda}\hat{f}(y).\end{aligned}$$

By the inversion formula, we have

$$\varphi(0) = \sum_{y\in\frac{1}{\lambda}\mathbb{Z}} \hat{\varphi}(y).$$

□

EXAMPLE 4 For $f(x) = e^{-\pi\|x\|^2}$, we obtain that for any $\lambda > 0$,

$$\sum_{j=-\infty}^{\infty} e^{-\pi\|\lambda j\|^2} = \frac{1}{\lambda}\sum_{j=-\infty}^{\infty} e^{-\pi\|\frac{j}{\lambda}\|^2}.$$

2.2 The n-dimensional case

In this subsection, we extend the definition of the Fourier series to the n-dimensional case. We start by considering the Fourier series of functions on $\mathbb{R}^n$ that are $\mathbb{Z}^n$-periodic, that is, functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $f(x+y) = f(x)$ for any $x \in \mathbb{R}^n, y \in \mathbb{Z}^n$.

DEFINITION 5 *For a $\mathbb{Z}^n$-periodic function f define its Fourier series $\hat{f} : \mathbb{Z}^n \to \mathbb{C}$ as*

$$\hat{f}(y) = \int_{[0,1)^n} f(x)e^{-2\pi i\langle x,y\rangle}dx$$

THEOREM 10 *For a nice enough f we have that for all x*

$$f(x) = \sum_{y\in\mathbb{Z}^n} \hat{f}(y)e^{2\pi i\langle x,y\rangle}$$

LEMMA 11 (PSF$_3$) *For a nice enough f we have $f(\mathbb{Z}^n) = \hat{f}(\mathbb{Z}^n)$*

We would now like to extend the above to functions that are Λ-periodic for some full-rank lattice Λ. Notice that we already did a similar thing in the previous subsection for one-dimensional lattices. Indeed, we started with $\mathbb{Z}$-periodic functions and then extended our discussion to Λ-periodic functions for any one-dimensional lattice Λ. The Fourier series of $\lambda\mathbb{Z}$-periodic functions was defined as a function on the dual lattice $\frac{1}{\lambda}\mathbb{Z}$. Moreover, in Lemma 9 we proved that $f(\Lambda) = \det(\Lambda^*) \cdot \hat{f}(\Lambda^*)$ for any one-dimensional lattice Λ.

Let B be a basis of some full-rank lattice Λ and let f be a Λ-periodic function, i.e., a function $f : \mathbb{R}^n \to \mathbb{C}$ such that $f(x+y) = f(x)$ for any $x \in \mathbb{R}^n, y \in \Lambda$. The Fourier series of f is the function $\hat{f} : \Lambda^* \to \mathbb{C}$ given by

$$\hat{f}(y) = \frac{1}{\det(\Lambda)}\int_{\mathcal{P}(B)} f(x)e^{-2\pi i\langle x,y\rangle}dx.$$

As the following exercise shows, this definition is independent of the choice of basis B, and is therefore well-defined.

EXERCISE 1 Show that if g is a Λ-periodic function for some lattice $\Lambda = \mathcal{L}(B)$, then its integral on $\mathcal{P}(B)$ is the same for any choice of basis B. (A possible approach is to show that the integral is invariant under the basic operations and then use the fact that a basis can be transformed into any other basis using a sequence of basic operations.) Deduce that $\hat{f}$ is well-defined.

The inversion formula is now of the following form.

THEOREM 12 *For a nice enough f we have that for all x*

$$f(x) = \sum_{z \in \Lambda^*} \hat{f}(z) e^{2\pi i \langle x, z \rangle}$$

Finally, we have the following general formulation of the Poisson summation formula. This formulation will be often used is future lectures.

LEMMA 13 (PSF$_4$) *For a nice enough f and any full-rank lattice Λ, $f(\Lambda) = \det(\Lambda^*)\hat{f}(\Lambda^*)$.*

PROOF: The function $\varphi(x) = \sum_{z \in \Lambda} f(x+z)$ is Λ-periodic and hence we can consider its Fourier series. For any $y \in \Lambda^*$ we have

$$\begin{aligned}
\hat{\varphi}(y) &= \frac{1}{\det(\Lambda)} \int_{\mathcal{P}(B)} \sum_{z \in \Lambda} f(x+z) e^{-2\pi i \langle x, y \rangle} dx \\
&= \frac{1}{\det(\Lambda)} \sum_{z \in \Lambda} \int_{\mathcal{P}(B)} f(x+z) e^{-2\pi i \langle x, y \rangle} dx \\
&= \frac{1}{\det(\Lambda)} \sum_{z \in \Lambda} \int_{\mathcal{P}(B)} f(x+z) e^{-2\pi i \langle x+z, y \rangle} dx \\
&= \frac{1}{\det(\Lambda)} \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, y \rangle} dx \\
&= \det(\Lambda^*) \hat{f}(y),
\end{aligned}$$

where we used that $\langle z, y \rangle \in \mathbb{Z}$. By the inversion formula, $\varphi(0) = \sum_{y \in \Lambda^*} \hat{\varphi}(y)$. $\square$

Let us remark that it is possible to derive Lemma 13 directly from Lemma 11 by using the fact that if $f : \mathbb{R}^n \to \mathbb{C}$ is a Λ-periodic function for some lattice Λ with basis B, then the function $g : \mathbb{R}^n \to \mathbb{C}$ given by $g(x) = f(Bx)$ is $\mathbb{Z}^n$-periodic.

EXAMPLE 5 Applying this to the function ρ defined in Example 3, we obtain that $\rho(\Lambda) = \det(\Lambda^*)\rho(\Lambda^*)$. More generally, we obtain $\rho_s(\Lambda) = s^n \det(\Lambda^*) \rho_{1/s}(\Lambda^*)$.