Homework 5
Due 2011/5/13

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Instructions as before.

1. **Stronger KKL theorem:** Prove the following strengthening of the KKL theorem. There exists a $c > 0$ such that if $f : \{0,1\}^n \to \{-1,1\}$ is a balanced function with $\mathrm{Inf}_i(f) \le \delta$ for all i, then $\mathbb{I}(f) \ge c \log(1/\delta)$.

2. **Talagrand's lemma:** Let $f : \{0,1\}^n \to [-1,1]$ and assume $p = \mathbb{E}[|f|] \ll 1$. Show that $W_1(f) = \sum_{|S|=1} \hat{f}(S)^2 \le O(p^2 \log(1/p))$.

3. **Generalized Chernoff bound:** Let $p(x_1,\ldots,x_n)$ be a multilinear polynomial over the reals of degree at most d, and assume that $\mathbb{E}[p(x_1,\ldots,x_n)^2] = 1$ where the x_i are chosen independently from $\{-1,1\}$ (equivalently, this says that the sum of squares of p's coefficients is 1). Then for any large enough t,
$$\Pr[|p(x_1,\ldots,x_n)| \ge t] \le \exp(-\Omega(t^{2/d})),$$
where the x_i are chosen as before. The case $d = 1$ is a version of the Chernoff bound. Hint: use Markov's inequality and a corollary of the hypercontractive inequality that we saw in class.

4. **Logarithmic Sobolev inequality:**
   (a) Using the hypercontractive inequality, show that for any $f : \{0,1\}^n \to \mathbb{R}$ and $0 \le \varepsilon \le \frac{1}{2}$,
   $$\|T_{\sqrt{1-2\varepsilon}} f\|_2^2 \le \|f\|_{2-2\varepsilon}^2.$$
   (b) Notice that we have equality at $\varepsilon = 0$ and use this to deduce
   $$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \|T_{\sqrt{1-2\varepsilon}} f\|_2^2 \Big|_{\varepsilon=0} \le \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \|f\|_{2-2\varepsilon}^2 \Big|_{\varepsilon=0}.$$
   (c) Show that the left hand side is $-2\mathbb{I}(f)$.
   (d) Show that the right hand side is $-\mathrm{Ent}[f^2]$ where $\mathrm{Ent}[g]$ is defined for non-negative g as $\mathbb{E}[g \ln g] - \mathbb{E}[g] \ln \mathbb{E}[g]$ (with $0 \ln 0$ defined as 0). No need to be 100% rigorous.

   This establishes the *logarithmic Sobolev inequality*, saying that for any $f : \{0,1\}^n \to \mathbb{R}$,
   $$\mathrm{Ent}[f^2] \le 2\mathbb{I}(f).$$
   (e) Show that if $f : \{0,1\}^n \to \{-1,1\}$ has $p = \Pr[f = -1] \le \frac{1}{2}$ then
   $$\mathbb{I}(f) \ge 2p \ln(1/p).$$
   For small value of p, this significantly improves the Poincaré inequality $\mathbb{I}(f) \ge 4p(1-p)$ from Homework 1.

5. **Talagrand's open question (\$1000):** Fix some $0 < \rho < 1$. Let $f : \{0,1\}^n \to [0,1]$ and let $\mu = \mathbb{E}[f]$. Note that $\mathbb{E}[T_\rho f] = \mu$ as well. Clearly, Markov's inequality implies that $\Pr[(T_\rho f)(x) \ge t\mu] \le \frac{1}{t}$. Can you improve this upper bound to $o(\frac{1}{t})$? perhaps $O(1/(t\sqrt{\log t}))$? Intuitively, since T_ρ smoothes f, one would expect the peaks to shrink.