Mathematical Theories of Turbulent Dynamical Systems

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Lecture 11 & 12: Mathematical Theories of Turbulent Dynamical Systems

1. Nontrivial turbulent dynamical systems with a Gaussian invariant measure

2. Exact equations for the mean and covariance of the fluctuations
   - Turbulent dynamical systems with nontrivial third-order moments
   - Statistical dynamics in the L-96 model and statistical energy conservation

3. A statistical energy conservation principle for turbulent dynamical systems
   - Details about deterministic triad energy conservation symmetry
   - A generalized statistical energy identity
   - Enhanced dissipation of the statistical mean energy, the statistical energy principle, and “Eddy viscosity”
Outline

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   - Enhanced dissipation of the statistical mean energy, the statistical energy principle, and “Eddy viscosity”
Turbulent systems with special forms of damping and noise

The system setup will be a finite-dimensional system of, $\mathbf{u} \in \mathbb{R}^N$, with linear dynamics, an energy preserving quadratic part, and random white noise forcing

$$\frac{d\mathbf{u}}{dt} = \mathcal{L} [\mathbf{u}] = B (\mathbf{u}, \mathbf{u}) + L\mathbf{u} - d\Lambda\mathbf{u} + \Lambda^{1/2}\sigma \dot{W} (t; \omega).$$

Properties in the operators

$$\mathbf{u} \cdot B (\mathbf{u}, \mathbf{u}) = 0, \quad \text{Energy Conservation},$$
$$\mathbf{u} \cdot L\mathbf{u} = 0, \quad \text{Skew Symmetry for } L,$$
$$\text{div}_{\mathbf{u}} (B (\mathbf{u}, \mathbf{u})) = 0, \quad \text{Liouville Property}.$$

The scalar $d$ and $\sigma$ satisfy $\sigma_{eq}^2 = \sigma^2 / 2d$, and $\Lambda \geq 0$ (positive-definite) controls the linear damping rate and the noise amplitude.
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The scalar $d$ and $\sigma$ satisfy $\sigma_{eq}^2 = \sigma^2 / 2d$, and $\Lambda \geq 0$ (positive-definite) controls the linear damping rate and the noise amplitude.
General setup of turbulent systems

\[
\frac{du}{dt} = \mathcal{L} [u] = B(u, u) + L u - d \Lambda u + \Lambda^{1/2} \omega \dot{W}(t; \omega).
\]

- Examples:
  - General 3 $\times$ 3 triad system model;
  - Truncated Burgers-Hopf (TBH) equation, and inviscid L-96;
  - Quasi-geostrophic flow with topography, mean flow in pseudo-energy metric;
  - Inviscid Boussinesq or modified shallow water equation.
Gaussian invariant measure

\[
\frac{d\mathbf{u}}{dt} = \mathcal{L} [\mathbf{u}] = B (\mathbf{u}, \mathbf{u}) + L \mathbf{u} - d\Lambda \mathbf{u} + \Lambda^{1/2} \sigma \dot{\mathbf{W}} (t; \omega). 
\]  

(1)

**Proposition**

Associated with SDE in (1), with the structural properties, then

- (1) has the Gaussian invariant measure defined as in \( p_{\text{eq}} \);
- The invariant measure has equipartition of energy in each component of \( \mathbf{u} \),

\[
p_{\text{eq}} = C_N^{-1} \exp \left( -\frac{1}{2} \sigma_{\text{eq}}^{-2} \mathbf{u} \cdot \mathbf{u} \right), \text{ with } \sigma_{\text{eq}}^2 = \sigma^2 / 2d.
\]

**Proof.** The Fokker-Planck Equation for (1), using the properties, is given by

\[
\frac{dp}{dt} = -\text{div}_u [(B (\mathbf{u}, \mathbf{u}) + L \mathbf{u}) p] + \text{div}_u (d\Lambda \mathbf{u} p) + \text{div}_u \left( \Lambda \frac{\sigma^2}{2} \nabla p \right)
\]

\[
= - (B (\mathbf{u}, \mathbf{u}) + L \mathbf{u}) \cdot \nabla p + \text{div}_u \left( d\Lambda \mathbf{u} p + \frac{\Lambda \sigma^2}{2} \nabla p \right).
\]

Insert \( p_{\text{eq}} \)

\[
d\Lambda \mathbf{u} p_{\text{eq}} + \frac{\Lambda \sigma^2}{2} \nabla p_{\text{eq}} = d\Lambda \mathbf{u} p_{\text{eq}} - \frac{\Lambda \sigma^2}{2} \sigma_{\text{eq}}^{-2} \mathbf{u} p_{\text{eq}} \equiv 0.
\]
Gaussian invariant measure

\[ \frac{d\mathbf{u}}{dt} = \mathcal{L}[\mathbf{u}] = B(u, u) + Lu - d\Lambda u + \Lambda^{1/2}\sigma \dot{W}(t; \omega). \]  

(1)

Proposition

Associated with SDE in (1), with the structural properties, then

- (1) has the Gaussian invariant measure defined as in \( p_{eq} \);
- The invariant measure has equipartition of energy in each component of \( \mathbf{u} \),

\[ p_{eq} = C_N^{-1} \exp \left(-\frac{1}{2} \sigma_{eq}^{-2} \mathbf{u} \cdot \mathbf{u} \right), \text{ with } \sigma_{eq}^2 = \frac{\sigma^2}{2}. \]

Proof. The Fokker-Planck Equation for (1), using the properties, is given by

\[ \frac{dp}{dt} = -\text{div}_u [(B(u, u) + Lu) p] + \text{div}_u (d\Lambda u p) + \text{div}_u \left( \frac{\Lambda \sigma^2}{2} \nabla p \right) \]

\[ = -(B(u, u) + Lu) \cdot \nabla p + \text{div}_u \left( d\Lambda u p + \frac{\Lambda \sigma^2}{2} \nabla p \right). \]

Insert \( p_{eq} \)

\[ d\Lambda u p_{eq} + \frac{\Lambda \sigma^2}{2} \nabla u p_{eq} = d\Lambda u p_{eq} - \frac{\Lambda \sigma^2}{2} \sigma_{eq}^{-2} u p_{eq} \equiv 0. \]
Example: a simplified 57-mode testing model

In the numerical experiments, the truncation $|k|^2 \leq \Lambda$, with $\Lambda = 17$ are utilized with 57 degrees of freedom

\[
\frac{d\hat{\omega}_k}{dt} = \left( \nabla^\perp \psi \cdot \nabla q \right)_k + ik_1 \left( \beta / |k|^2 - V \right) \hat{\omega}_k - ik_1 \hat{h}_k V
\]

\[-d\lambda_k \omega_k + \sigma \left( 1 + \mu / |k|^2 \right)^{-1/2} \lambda_k^{1/2} \hat{W}_k,
\]

\[
\frac{dV}{dt} = \sum_{1 \leq |k| \leq \Lambda} \frac{ik_1}{|k|^2} \hat{h}_k^* \hat{\omega}_k - d\lambda_0 V + \sigma \mu^{-1/2} \lambda_0^{1/2} \hat{W}_0.
\]

The pseudo-energy metric induces

\[
p_{eq}(V, \omega) = C^{-1} \exp \left[ -\frac{\mu}{2} \sigma_{eq}^{-2} (V - \bar{V})^2 - \frac{1}{2} \sigma_{eq}^{-1} \sum_{1 \leq |k|^2 \leq \Lambda} \left( 1 + \frac{\mu}{|k|^2} \right) (\omega_k - \bar{\omega}_k)^2 \right].
\]

So the steady state large and small scale variables reach the equilibrium mean and variance

\[
\bar{V} = -\frac{\beta}{\mu}, \quad \bar{\psi}_k = \frac{h_k}{\mu + |k|^2}, \quad |V'|^2 = \frac{\sigma_{eq}^2}{\mu}, \quad |\psi'|^2 = \frac{\sigma_{eq}^2}{|k|^2 \left( |k|^2 + \mu \right)}.
\]

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1 Majda, Timofeyev, and Vanden-Eijnden, JAS, 2003
2 Majda, Franzke, Fischer, and Crommelin, PNAS, 2006
Equilibrium steady state data

|       | $\bar{V}$ | $V'^2$ | $\bar{\psi}_{(1,0)}$ | $|\psi'|^2_{(1,0)}$ |
|-------|-----------|--------|----------------------|--------------------|
| theory| -0.5      | 0.5    | -0.1768+0.1768i      | 0.3333             |
| numerics| -0.5228   | 0.5080 | -0.1764+0.1799i      | 0.3353             |

|       | $|\psi'|^2_{(0,1)}$ | $|\psi'|^2_{(2,0)}$ | $|\psi'|^2_{(0,2)}$ | $V'\psi'_{(1,0)}$ |
|-------|---------------------|---------------------|---------------------|-------------------|
| theory| 0.3333              | 0.0417              | 0.0417              | 0                 |
| numerics| 0.3350             | 0.0414              | 0.0421              | $(2.5+1.8i) \times 10^{-3}$ |

Table: Unperturbed equilibrium statistics of the 57-mode model with damping and forcing term. The parameters $\mu = 2$, $\sigma^2_{eq} = 1$, $\beta = 1$, $H = 3\sqrt{2}/4$, $d = 0.5$ are used in the model.
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General setup of turbulent systems with quadratic nonlinearities

The system setup will be a finite-dimensional system with linear dynamics and an energy preserving quadratic part with \( u \in \mathbb{R}^N \)

\[
\frac{du}{dt} = \mathcal{L} [u(t; \omega); \omega] = (L + D)u + B(u, u) + F(t) + \sigma_k(t) \dot{W}_k(t; \omega), \quad (3)
\]

\[
u(t_0; \omega) = u_0(\omega). \quad (4)
\]

- \( L \) being a skew-symmetric linear operator \( L^* = -L \), representing the \( \beta \)-effect of Earth’s curvature, topography etc.
- \( D \) being a negative definite symmetric operator \( D^* = D \), representing dissipative processes such as surface drag, radiative damping, viscosity etc.
- \( B(u, u) \) being a quadratic operator which conserves the energy by itself so that it satisfies \( B(u, u) \cdot u = 0 \).
- \( F(t) + \sigma_k(t) \dot{W}_k(t; \omega) \) being the effects of external forcing, i.e. solar forcing, seasonal cycle, which can be split into a mean component \( F(t) \) and a stochastic component with white noise characteristics.
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Ideas about Reduced Order Methods (ROMs)

Approach:

- Project dynamics onto a finite-dimensional representation of the stochastic field consisting of fixed-in-time, $N$-dimensional, orthonormal basis $\{v_i\}_{i=1}^N$

$$u(t) = \bar{u}(t) + u'(t), \quad u' = \sum_{i=1}^{N} \Phi_i(t; \omega) v_i \approx \sum_{i=1}^{s} \tilde{\Phi}_i(t; \omega) v_i;$$

- Modeling is restricted to subspace $V_s = \text{span} \{v_i\}_{i=1}^{s}$;
- Aim is to capture as much of the dynamics as possible with few modes;
- Attractive for reduced computational cost.
Finite-dimensional representation of the stochastic field

Use a finite-dimensional representation of the stochastic field consisting of fixed-in-time, $N$-dimensional ($u \in \mathbb{R}^N$), orthonormal basis $\{v_i\}$

$$u(t) = \bar{u}(t) + u'(t) = \bar{u}(t) + \sum_{i=1}^{N} Z_i(t;\omega) v_i, \quad \bar{u}(t) = \langle u(t) \rangle, \quad \langle u' \rangle = 0.$$ 

Constructing the random field $v_i$ from equilibrium statistics

$$C_{uu}(x,y) = \mathbb{E}_\omega [(u(x;\omega) - \bar{u}(x;\omega))(u(y;\omega) - \bar{u}(y;\omega))^*]$$

Solve the eigenvalue problem

$$\int C_{uu}(x,y) v_i dx = \lambda_i^2 v_i(y).$$

- Provides principal directions $v_i$ spanning the phase space over which probability is distributed.
- For each principal direction $v_i$, the associated eigenvalue $\lambda_i$ defines the spread of the probability.
Finite-dimensional representation of the stochastic field

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- Provides principal directions $v_i$ spanning the phase space over which probability is distributed
- For each principal direction $v_i$ the associated eigenvalue $\lambda_i$ defines the spread of the probability
Development of statistical moment dynamics

\[ u(t) = \bar{u}(t) + u'(t) = \bar{u}(t) + \sum_{i=1}^{N} Z_i(t; \omega) v_i, \quad \langle u' \rangle = 0. \]

Take expectation on both sides of the equations

\[ \frac{du}{dt} = (L + D) u + B(u, u) + F(t) + \sigma_k(t) \dot{W}_k(t; \omega) \quad (5) \]

Equation for the mean field \( \bar{u} \)

\[ \frac{d\bar{u}}{dt} = (L + D) \bar{u} + B(\bar{u}, \bar{u}) + R_{ij} B(v_i, v_j) + F(t), \quad (6) \]

where \( R_{ij} = \langle Z_i Z_j^* \rangle \),

\[ B(u, u) = B(\bar{u} + u', \bar{u} + u') = B(\bar{u}, \bar{u}) + B(\bar{u}, u') + B(u', \bar{u}) + B(u', u'), \]

\[ \langle B(u', u') \rangle = \langle B(Z_i v_i, Z_j v_j) \rangle = \langle Z_i Z_j^* \rangle B(v_i, v_j). \]
Equation for the fluctuation component

\[
\frac{du'}{dt} = (L + D) u' + B (\bar{u}, u') + B (u', \bar{u}) + B (u', u') - R_{jk} B (v_j, v_k) + \sigma_k(t) \dot{W}_k(t; \omega).
\] (7)

Equations for the stochastic coefficients

\[
\frac{dZ_i}{dt} = Z_j [(L + D) v_j + B (\bar{u}, v_j) + B (v_j, \bar{u})] \cdot v_i + [B (u', u') - R_{jk} B (v_j, v_k)] \cdot v_i + \dot{W}_k \sigma_k \cdot v_i.
\] (8)

Equation for the covariance matrix \( R_{ij} = \langle Z_i Z_j^* \rangle \)

\[
\frac{dR}{dt} = L_v (\bar{u}) R + RL_v^* (\bar{u}) + Q_F + Q_\sigma.
\] (9)
Covariance Equation

\[
\frac{dR}{dt} = L_v (\bar{u}) R + RL_v^* (\bar{u}) + Q_F + Q_\sigma
\]

- the linear dynamics operator \( L_v \) expressing energy transfers between the **mean field and the stochastic modes** (\( B \)), as well as energy dissipation (\( D \)), and non-normal dynamics (\( L \))
  \[
  \{L_v\}_{ij} = [(L + D) \mathbf{v}_j + B (\bar{u}, \mathbf{v}_j) + B (\mathbf{v}_j, \bar{u})] \cdot \mathbf{v}_i.
  \]

- the positive definite operator \( Q_\sigma \) expressing energy transfer due to external stochastic forcing
  \[
  \{Q_\sigma\}_{ij} = \mathbf{v}_i^* \sigma_k^* \sigma_k \mathbf{v}_j.
  \]

- the third-order moments expressing the energy flux between different modes due to non-linear terms
  \[
  Q_F = \langle Z_m Z_n Z_i \rangle B (\mathbf{v}_m, \mathbf{v}_n) \cdot \mathbf{v}_i + \langle Z_m Z_n Z_i \rangle B (\mathbf{v}_m, \mathbf{v}_n) \cdot \mathbf{v}_j.
  \]

**Note** that energy is still conserved in this nonlinear interaction part

\[
\text{Tr} [Q_F] = 2 \langle Z_m Z_n Z_i \rangle B (\mathbf{v}_m, \mathbf{v}_n) \cdot \mathbf{v}_i
\]
\[
= 2 \langle B (Z_m \mathbf{v}_m, Z_n \mathbf{v}_n) \cdot Z_i \mathbf{v}_i \rangle = 2 \langle B (\mathbf{u}', \mathbf{u}') \cdot \mathbf{u}' \rangle = 0.
\]
Energy flow in the quadratic systems

Statistical mean and covariance dynamics, \( \mathbf{u} = \bar{\mathbf{u}} + Z_i \mathbf{v}_i \), \( R_{ij} = \langle Z_i Z_j^* \rangle \),

\[
\begin{align*}
\frac{d\bar{\mathbf{u}}}{dt} &= (L + D) \bar{\mathbf{u}} + B(\bar{\mathbf{u}}, \bar{\mathbf{u}}) + R_{ij}B(\mathbf{v}_i, \mathbf{v}_j) + \mathbf{F}(t), \\
\frac{dR}{dt} &= L^*_\mathbf{v}(\bar{\mathbf{u}}) R + R L^*_\mathbf{v}(\bar{\mathbf{u}}) + Q_F + Q_\sigma.
\end{align*}
\]

\[
Q_F = \langle Z_m Z_n Z_j \rangle B(\mathbf{v}_m, \mathbf{v}_n) \cdot \mathbf{v}_i + \langle Z_m Z_n Z_i \rangle B(\mathbf{v}_m, \mathbf{v}_n) \cdot \mathbf{v}_j
\]
Consider a turbulent dynamical system without noise, \( \sigma \equiv 0 \), and assume it has a statistical steady state so that \( \bar{u}_{eq} \) and \( R_{eq} \) are time independent. Since \( \frac{d}{dt} R_{eq} = 0 \), \( R_{eq} \) necessarily satisfies the steady covariance equation (3.6)

\[
L\bar{u}_{eq} R_{eq} + R_{eq} L^* \bar{u}_{eq} = -Q_{F,eq},
\]

(3.11)

where \( Q_{F,eq} \) are the third moments from (3.9) evaluated at the statistical steady state. Thus a necessary and sufficient condition for a non-Gaussian statistical steady state is that the first and second moments satisfy the obvious requirement that the matrix

\[
L\bar{u}_{eq} R_{eq} + R_{eq} L^* \bar{u}_{eq} \neq 0,
\]

(3.12)

So the above matrix has some non-zero entries. The non-trivial third moments play a crucial dynamical role in the L-96 model and for two-layer ocean turbulence.
Turbulent dynamical systems with non-Gaussian statistical steady states

In the system with a Gaussian invariant measure

\[ \frac{du}{dt} = B(u, u) + Lu - d\Lambda u + \Lambda^{1/2} \sigma \dot{W}(t; \omega). \]

- There \( \bar{u}_{eq} = 0 \), \( R_{eq} = \sigma^2_{eq} I \), and \( L\bar{u}_{eq} = L \) with \( L \) skew-symmetric, so

  \[ LR_{eq} + R_{eq} L^* \equiv 0; \]

- The damping with matrix \( D = -d\Lambda \) exactly balances the stochastic forcing variance \( \sigma^2\Lambda \);

- For a strictly positive covariance \( R_{eq} \), there is a “whitening” linear transformation, so that \( TR_{eq} T^{-1} = I \). Nontrivial third moments are satisfied if

  \[ TL_{\bar{u}_{eq}} T^{-1}, \]

is non-zero.
Lorenz ’96 system

L-96

\[
\frac{du_i}{dt} = u_{i-1}(u_{i+1} - u_{i-2}) - d_i(t)u_i + F_i(t), \quad i = 0, 1, \ldots, J-1, \quad J = 40.
\]

\[ u_J = u_0. \]

![Diagram of space-time of numerical solutions of L-96 model for weakly chaotic (F = 6), strongly chaotic (F = 8), and fully turbulent (F = 16) regime.]{fig2.1}

2.3 Statistical Triad Models, the Building Blocks of Complex Turbulent Dynamical Systems

Statistical triad models are special three dimensional turbulent dynamical systems with quadratic nonlinear interactions that conserve energy. For \( u = (u_1, u_2, u_3)^T \in \mathbb{R}^3 \), these equations can be written in the form of (1.5)–(1.7) with a slight abuse of notation as

\[
\frac{du}{dt} = Lu + Du + B(u, u) + F + s \dot{W}_t, \quad (2.4)
\]
Statistical formulation of the L-96 system

We use a Fourier basis

\[ v_k = \left\{ \frac{1}{\sqrt{J}} e^{2\pi i j k} \right\}_{j=0}^{J-1}, \]

because they diagonalize translation invariant systems with spatial homogeneity. Here are the \textbf{statistical dynamics for L-96 model}:

\[ \frac{d\bar{u}(t)}{dt} = -d(t)\bar{u}(t) + \frac{1}{J} \sum_{k=-J/2+1}^{J/2} r_k(t) \Gamma_k + F(t) \]

\[ \frac{dr_k(t)}{dt} = 2 \left[ -\Gamma_k \bar{u}(t) - d(t) \right] r_k(t) + Q_{F,kk}, \quad k = 0, 1, ..., J/2. \]

Here we denote \( \Gamma_k = \cos \frac{4\pi k}{J} - \cos \frac{2\pi k}{J} \), \( r_{-k} = \langle Z_{-k} Z^*_{-k} \rangle = \langle Z_k Z^* \rangle = r_k \), and the nonlinear flux \( Q_F \) for the third moments becomes diagonal

\[ Q_{F, kk'} = \frac{2}{\sqrt{J}} \sum_m \Re \left\{ \langle Z_m Z_{-m-k} Z_k \rangle \left( e^{-2\pi i \frac{2m+k}{J}} - e^{2\pi i \frac{m+2k}{J}} \right) \right\} \delta_{kk'}, \]

with energy conservation \( \text{tr} Q_F = 0 \).
Statistical energy conservation principle for L-96 model

By multiplying $\bar{u}$ on both sides of the mean equation and by summing up all the modes in the variance equation in (10), we get

$$\frac{d\bar{u}^2}{dt} = -2d\bar{u}^2 + 2\bar{u}F + \frac{2}{J} \sum_k \Gamma_k r_k \bar{u}$$

$$\frac{d\text{tr} R}{dt} = 2 \left(-\sum_k \Gamma_k r_k \bar{u}\right) - 2d\text{tr} R + \text{tr} Q_F.$$

It is convenient to define the **statistical energy** including both mean and total variance as

$$E(t) = \frac{J}{2} \bar{u}^2 + \frac{1}{2} \text{tr} R.$$ (11)

With this definition the corresponding dynamical equation for the statistical energy $E$ of the true system can be easily derived as

$$\frac{dE}{dt} = -2dE + JF \bar{u} + \frac{1}{2} \text{tr} Q_F = -2dE + JF \bar{u},$$ (12)

with symmetry of nonlinear energy conservation, $\text{tr} Q_F = 0$, assumed.
Nontrivial third moments for statistical steady state of L-96 model

Consider the original L-96 model with constant forcing $F$ and damping $d = 1$ with the statistical steady states $\bar{u}_{eq}, R_{eq}$, with $R_{eq} = \{r_{eq,i}\delta_{ij}\}$ diagonal. For the L-96 model with homogeneous statistics, the linear operator is also block diagonal with the symmetric part $L^s_{u_{eq}} = \{L^s_{u_{eq},i}\delta_{ij}\}$ and

$$L^s_{u_{eq},i} = -\Gamma_i \bar{u}_{eq} - 1, \ i = 0, \cdots, J/2.$$ 

The third moments are non-zero and the statistical steady state is non-Gaussian provided

$$L^s_{u_{eq},i} r_{eq,i} \neq 0, \ \text{for some } i \ \text{with } 0 \leq i \leq \frac{J}{2},$$

in which case

$$-2L^s_{u_{eq},i} r_{eq,i} = Q_{F,eq,i}, \ 0 \leq i \leq \frac{J}{2}.$$
Nontrivial third moments for statistical steady state of L-96 mode

(a) third order moments between modes in spectral domain

(b) third order moments between state variables at different physical grid points
Nontrivial turbulent dynamical systems with a Gaussian invariant measure

Exact equations for the mean and covariance of the fluctuations
- Turbulent dynamical systems with nontrivial third-order moments
- Statistical dynamics in the L-96 model and statistical energy conservation

A statistical energy conservation principle for turbulent dynamical systems
- Details about deterministic triad energy conservation symmetry
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A general statistical energy principle

- In L-96 system with homogeneous statistics, we have the statistical energy dynamics for the total statistical energy $E(t) = \frac{1}{2} \bar{u}^2 + \frac{1}{2} \text{tr} R$

$$\frac{dE}{dt} = -2dE + JF\bar{u}.\$$

- A fundamental issue in UQ is that complex turbulent dynamical systems are highly anisotropic with inhomogeneous forcing and the statistical mean can exchange energy with the fluctuations.

- There is suitable statistical symmetry so that the energy of the mean plus the tracer of the covariance matrix satisfies an energy conservation principle for the general conservative nonlinear dynamics

$$\frac{d\bar{u}}{dt} = (L + D)\bar{u} + B(\bar{u}, \bar{u}) + F(t) + \sigma_k(t)\dot{W}_k(t; \omega),$$

where exact mean statistical field equation and the covariance equation can be calculated as

$$\frac{d\bar{u}}{dt} = (L + D)\bar{u} + B(\bar{u}, \bar{u}) + R_{ij}B(e_i, e_j) + F,$$

$$\frac{dR}{dt} = \nu L R + RL^* + Q_F + Q_\sigma.$$
Detailed symmetry in triad energy conservation

Proposition

Consider the three dimensional Galerkin projected dynamics spanned by the triad $e_i, e_j, e_k$ for $1 \leq i, j, k \leq N$ for the pure nonlinear model

$$(u_\wedge)_t = \mathcal{P}_\wedge B (u_\wedge, u_\wedge).$$

(13)

Assume the following:

A) The self interactions vanish,

$$B (e_i, e_i) \equiv 0, \quad 1 \leq i \leq N;$$

(14)

B) The dyad interaction coefficients vanish through the symmetry,

$$e_i \cdot [B (e_i, e_i) + B (e_i, e_l)] = 0, \quad \text{for any } i, l.$$ (15)
Proposition

(continue) Then the three dimensional Galerkin truncation becomes the triad interaction equations for \( u = (u_i, u_j, u_k) = (u_\Lambda \cdot e_i, u_\Lambda \cdot e_j, u_\Lambda \cdot e_k) \)

\[
\begin{align*}
\frac{du_i}{dt} &= A_{ijk} u_j u_k, \\
\frac{du_j}{dt} &= A_{jki} u_k u_i, \\
\frac{du_k}{dt} &= A_{kij} u_i u_j.
\end{align*}
\]

(16)

with coefficient satisfying

\[
A_{ijk} + A_{jik} + A_{kji} = 0,
\]

(17)

which is the detailed triad energy conservation symmetry, since

\[
\begin{align*}
A_{ijk} + A_{jki} + A_{kij} & \equiv e_i \cdot [B(e_j, e_k) + B(e_k, e_j)] + \\
e_j \cdot [B(e_k, e_i) + B(e_i, e_k)] + \\
e_k \cdot [B(e_i, e_j) + B(e_j, e_i)] \\
= 0.
\end{align*}
\]
The projected energy conservation law

Define $\Lambda$ as the index set for the resolved modes under the orthonormal basis $\{e_i\}_{i=1}^N$ of full dimensionality $N$ in the truncation model

$$u_\Lambda = \mathcal{P}_\Lambda u = \sum_{i \in \Lambda} u_i e_i.$$ 

The projected energy conservation law for truncated energy $E_\Lambda = \frac{1}{2}u_\Lambda \cdot u_\Lambda$ is satisfied depending on the proper conserved quantity and the induced inner product

$$\frac{dE_\Lambda}{dt} = u_\Lambda \cdot \mathcal{P}_\Lambda B (u_\Lambda, u_\Lambda) = 0.$$ \hspace{1cm} (18)

The second equality holds since

$$u_\Lambda \cdot \mathcal{P}_\Lambda B (u_\Lambda, u_\Lambda) = u_\Lambda \cdot \sum_{i \in \Lambda} [e_i \cdot B (u_\Lambda, u_\Lambda)] e_i$$

$$= \left( \sum_{j \in \Lambda} u_j e_j \right) \cdot \sum_{i=1}^N [e_i \cdot B (u_\Lambda, u_\Lambda)] e_i$$

$$= u_\Lambda \cdot B (u_\Lambda, u_\Lambda),$$
Proof of the Proposition

Now consider the triad truncated system about state variable $u_\Lambda$ in a three dimensional subspace. We take the index set with resolved modes as $\Lambda = \{i, j, k\}$. the explicit expressions can thus be calculated along each mode $e_m, m = i, j, k$ by applying assumption A) and B) in the Proposition

$$\frac{du_m}{dt} = e_m \cdot \mathcal{P}_\Lambda B (u_\Lambda, u_\Lambda) = e_m \cdot B \left( \sum_{n \in \Lambda} u_n e_n, \sum_{l \in \Lambda} u_l e_l \right) = \sum_{n, l \in \Lambda} u_n u_l e_m \cdot B (e_n, e_l) = \sum_{n \neq l \in \Lambda - \{m\}} u_n u_l e_m \cdot B (e_n, e_l) = u_n u_l e_m \cdot [B (e_n, e_l) + B (e_l, e_n)], \quad n \neq l \neq m.
Proof of the Proposition

The interaction coefficients therefore can be defined as

\[ A_{ijk} = e_i \cdot [B(e_j, e_k) + B(e_k, e_j)], \]

with symmetry

\[ A_{ijk} = A_{ikj}, \]

and vanishing property

\[ A_{ijk} = 0, \text{ if two of the index } i, j, k \text{ coincident.} \]

With this explicit definition of the coefficients \( A_{ijk} \), the detailed triad energy conservation symmetry is just direct application of the above formulas, that is for any \( u_i, u_j, u_k \) with \( i \neq j \neq k \)

\[
\frac{dE^\wedge}{dt} = u_i \frac{du_i}{dt} + u_j \frac{du_j}{dt} + u_k \frac{du_k}{dt} = (A_{ijk} + A_{jki} + A_{kij}) u_i u_k u_j = 0.
\]
Dynamics for the mean and fluctuation energy

**Proposition**

The change of mean energy $\bar{E} = \frac{1}{2} (\bar{u} \cdot \bar{u})$ satisfies

$$
\frac{d}{dt} \left( \frac{1}{2} |\bar{u}|^2 \right) = \bar{u} \cdot D\bar{u} + \bar{u} \cdot F + \frac{1}{2} R_{ij} \bar{u} \cdot [B(e_i, e_j) + B(e_j, e_i)].
$$

(19)

The last term represents the effect of the fluctuation on the mean, $\bar{u}$.

**Proposition**

Under the structure assumption in (14) and (15) on the basis $e_i$, the fluctuating energy, $E' = \frac{1}{2} \text{tr} R$, for any turbulent dynamical system satisfies,

$$
\frac{dE'}{dt} = \frac{1}{2} \text{tr} (\tilde{D}R + R\tilde{D}^*) + \frac{1}{2} \text{tr} Q_\sigma
$$

$$
-\frac{1}{2} R_{ij} \bar{u} \cdot [B(e_i, e_j) + B(e_j, e_i)],
$$

(20)

where $R$ satisfies the exact covariance equation.
Theorem

(Statistical Energy Conservation Principle) Under the structural assumption (14), (15) on the basis $e_\epsilon$, for any turbulent dynamical systems in the form

$$\frac{du}{dt} = (L + D)u + B(u, u) + F(t) + \sigma_k(t) \dot{W}_k(t; \omega),$$

the total statistical energy, $E = \bar{E} + E' = \frac{1}{2} \bar{u} \cdot \bar{u} + \frac{1}{2} \text{tr} R$, satisfies

$$\frac{dE}{dt} = \bar{u} \cdot D\bar{u} + \bar{u} \cdot F + \text{tr} \left( \tilde{D}R \right) + \frac{1}{2} \text{tr} Q_{\sigma}, \quad (21)$$

where $R$ satisfies the exact covariance equation.

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$^3$Majda, Statistical energy conservation principle for inhomogeneous turbulent dynamical systems, PNAS, 2015.
Illustration about the proof

\( \hat{R} \) is the tensor representation of the covariance matrix with \( u' = Z_i e_i \),

\[
\hat{R} = \langle u' \otimes u' \rangle = \sum_{i,j} R_{ij} e_i \otimes e_j;
\]

the matrix \( B \left( \bar{u}, \hat{R} \right) \) is defined as the componentwise interaction with each column of \( \hat{R} \).

Expand \( \bar{u} \) by \( \bar{u} = \bar{u}_M e_M \), and use \( \text{tr} (a \otimes b) = a \cdot b \), we have

\[
\frac{1}{2} \text{tr} \left[ B \left( \bar{u}, \hat{R} \right) + B \left( \hat{R}, \bar{u} \right) + 'transpose part' \right] = \frac{1}{2} R_{ij} \bar{u}_M \left[ e_i \cdot B (e_M, e_j) + B (e_i, e_M) \cdot e_j + e_j \cdot B (e_M, e_i) + B (e_j, e_M) \cdot e_i \right].
\]

Now use the detailed triad conservation structure, the formula is given by

\[
e_i \cdot \left[ B (e_j, e_M) + B (e_M, e_j) \right] + e_j \cdot \left[ B (e_M, e_i) + B (e_i, e_M) \right] = -e_M \cdot \left[ B (e_i, e_j) + B (e_j, e_i) \right].
\]

Now, use \( \bar{u} = \bar{u}_M e_M \) and get that the sum in (17) for this contribution to the trace is exactly

\[
- \frac{1}{2} R_{ij} \bar{u} \cdot \left[ B (e_i, e_j) + B (e_j, e_i) \right].
\]
Illustrative general examples and applications

Corollary

Under the assumption of the Theorem, assume $D = -dl$, with $d > 0$, then the turbulent dynamical system satisfies the closed statistical energy equation for $E = \frac{1}{2} \bar{u} \cdot \bar{u} + \frac{1}{2} \text{tr} R$,

$$\frac{dE}{dt} = -2dE + \bar{u} \cdot F + \frac{1}{2} \text{tr} Q_\sigma. \quad (22)$$

In particular, if the external forcing vanishes so that $F \equiv 0, Q_\sigma \equiv 0$, for random initial conditions, the statistical energy decays exponentially in time and satisfies $E(t) = \exp(-2dt) E_0$.

Assume the symmetric dissipation matrix, $D$, satisfies the upper and lower bounds,

$$-d_+ |u|^2 \geq u \cdot Du \geq -d_- |u|^2,$$

with $d_-, d_+ > 0$. Typical general dissipation matrices $\tilde{D}$ are diagonal in basis with Fourier modes or spherical harmonics, any positive symmetric matrix $R \geq 0$ we have the a priori bounds,

$$-d_+ \text{tr} R \geq \text{tr} \left( \frac{\tilde{D}R + R\tilde{D}^*}{2} \right) \geq -d_- \text{tr} R.$$
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Under the assumption of the Theorem, assume $D = -dl$, with $d > 0$, then the turbulent dynamical system satisfies the closed statistical energy equation for $E = \frac{1}{2} \bar{u} \cdot \bar{u} + \frac{1}{2} \text{tr} R$,

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Assume $\tilde{D}$ is diagonal and satisfies the upper and lower bounds, then the statistical energy in (21) in the Theorem, $E(t)$, satisfies the upper and lower bounds $E_+(t) \geq E(t) \geq E_-(t)$ where $E_{\pm}(t)$ satisfy the differential equality in Corollary 1 with $d \equiv d_{\pm}$. In particular, the statistical energy is a statistical Lyapunov function for the turbulent dynamical system. Also, if the external forcings $F$, $Q_{\sigma}$ vanish, the statistical energy decays exponentially with these upper and lower bounds.

Consider the Gaussian approximation to the one point statistics; recall that $u = \bar{u} + Z_i e_i$ so at the location $x$, the mean and variance are given by

$$\bar{u}(x) = \bar{u}_M e_M(x), \quad \text{var}(u(x)) = \langle Z_j Z_k^* \rangle e_j(x) \otimes e_k(x).$$

We have control over the variance of the average over the domain, denoted by $E_x$ because $E_x e_j(x) \otimes e_k(x) = \delta_{jk} I$; thus, the average of the single point variance is bounded by $\text{tr} R$ which is controlled by $E$. 
Corollary

Assume $\tilde{D}$ is diagonal and satisfies the upper and lower bounds, then the statistical energy in (21) in the Theorem, $E(t)$, satisfies the upper and lower bounds $E_+(t) \geq E(t) \geq E_-(t)$ where $E_\pm(t)$ satisfy the differential equality in Corollary 1 with $d \equiv d_\pm$. In particular, the statistical energy is a statistical Lyapunov function for the turbulent dynamical system. Also, if the external forcings $F, Q_\sigma$ vanish, the statistical energy decays exponentially with these upper and lower bounds.

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Example with turbulent flow

The nonlinear terms define $B$ with dynamics alone

$$\frac{\partial \omega_\Lambda}{\partial t} = -P_\Lambda (\nabla^\perp \psi_\Lambda \cdot \nabla \omega_\Lambda), \quad \triangle \psi_\Lambda = \omega_\Lambda,$$

which satisfy both the conservation of Energy, $E_\Lambda$, and Enstrophy, $E_\Lambda$ with

$$E_\Lambda = -\frac{1}{2} \int \psi_\Lambda \omega_\Lambda d\mathbf{x}, \quad \text{and} \quad E_\Lambda = \frac{1}{2} \int \omega_\Lambda^2 d\mathbf{x}.$$

It is well-known that in a Fourier basis, the symmetry conditions of Proposition are satisfied for either the $L^2$ inner product for $E_\Lambda$ or with a rescaled version of $E_\Lambda$

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle _E \equiv \int \omega_1 \omega_2 d\mathbf{x} = \int \triangle \psi_1 \triangle \psi_2 d\mathbf{x},$$

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle _E \equiv -\int \psi_1 \omega_2 d\mathbf{x} = \int \nabla \psi_1 \cdot \nabla \psi_2 d\mathbf{x},$$
Example with a dyad model

To check the **generalized energy principle**, we consider the simplest **dyad interaction equation**

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \gamma_1 u_1 u_2 + \gamma_2 u_2^2, \\
\frac{\partial u_2}{\partial t} &= -\gamma_1 u_1^2 - \gamma_2 u_1 u_2.
\end{align*}
\]

Take the two-dimensional natural basis \( e_1 = (1, 0)^T \) and \( e_2 = (0, 1)^T \), and the inner product is defined as the standard Euclidean inner product. Then the conservative quadratic interaction becomes

\[
B (e_1, e_1) = (0, -\gamma_1)^T ,
\]

\[
B (e_1, e_2) = B (e_2, e_1) = \left( \frac{1}{2} \gamma_1, -\frac{1}{2} \gamma_2 \right)^T ,
\]

\[
B (e_2, e_2) = (\gamma_2, 0)^T .
\]
• The energy conservation $\mathbf{u} \cdot B (\mathbf{u}, \mathbf{u}) = 0$ is satisfied in this model.

• Obviously the assumptions in (14) and (15) become non-zero

\[ B (\mathbf{e}_i, \mathbf{e}_i) \neq 0, \quad \mathbf{e}_i \cdot [B (\mathbf{e}_l, \mathbf{e}_i) + B (\mathbf{e}_i, \mathbf{e}_l)] \neq 0. \]

• On the other hand, the dyad interaction balance is satisfied so that

\[ \mathbf{e}_1 \cdot B (\mathbf{e}_1, \mathbf{e}_2) + \mathbf{e}_1 \cdot B (\mathbf{e}_2, \mathbf{e}_1) + \mathbf{e}_2 \cdot B (\mathbf{e}_1, \mathbf{e}_1) = \gamma_1 - \gamma_1 = 0. \]

Therefore the statistical energy conservation law is still valid for this dyad system

\[
\frac{d}{dt} \left[ \frac{1}{2} (\bar{u}_1^2 + \bar{u}_2^2) + \frac{1}{2} (\bar{u}_1^2 + \bar{u}_2^2) \right] = 0.
\]
Questions & Discussions