Equilibrium Statistical Mechanics for the Truncated Burgers-Hopf Equations

Di Qi, and Andrew J. Majda

Courant Institute of Mathematical Sciences

Fall 2016 Advanced Topics in Applied Math
Introduction

Consider large non-linear systems of ordinary differential equations

\[
\frac{d\vec{X}}{dt} = \vec{F}(\vec{X}), \quad \vec{X} \in \mathbb{R}^N, \quad \vec{F} = (F_1, \ldots, F_N), \quad N \gg 1,
\]

\[
\vec{X} \mid_{t=0} = \vec{X}_0.
\]

Common features of these systems:

- The dynamics are highly chaotic, thus a single trajectory is highly unpredictable;
- Observations as well as physical and numerical experiments indicate the existence of coherent patterns;
- It makes sense to study the ensemble behaviour of trajectories rather than a single trajectory.
Overview about equilibrium statistical mechanics

We study the ensembles of solutions in *probability measures* or *statistical solutions* on the phase space $\mathbb{R}^N$.

- **The Liouville property** is equivalent to volume preserving flow map. If an ensemble of initial data in the phase space is stretched in some direction, it must be compensated for by squeezing in some other direction under the flow map and vice versa.

- **The existence of conserved quantities** offer constraints on the probability distributions. The evolution of the probability measures on phase space enables us to define the associated information-theoretic entropy which sets the stage for the maximum entropy principle.

- **The most probable states (probability distributions), or the Gibbs measure** is obtained with the Liouville property and conserved quantities.
  - The Gibbs measure turns out to be an invariant measure of the flow maps, or a stationary statistical solution;
  - The dynamics is chaotic and instability is abundant so that there is some sort of ergodicity of the system$^1$.

---

Lecture 8: Equilibrium statistical mechanics for the truncated Burgers-Hopf equations

1. Review: Introduction to statistical mechanics for ODEs

2. Statistical mechanics for the truncated Burgers-Hopf equations
   - The truncated Burgers-Hopf system
   - The Gibbs measure and the prediction of equipartition of energy
   - Numerical evidence of the validity of the statistical theory
   - TBH as a model with statistical features with atmosphere

3. Statistically relevant conserved quantities for the truncated Burgers-Hopf equation
Outline

1 Review: Introduction to statistical mechanics for ODEs

2 Statistical mechanics for the truncated Burgers-Hopf equations
   - The truncated Burgers-Hopf system
   - The Gibbs measure and the prediction of equipartition of energy
   - Numerical evidence of the validity of the statistical theory
   - TBH as a model with statistical features with atmosphere

3 Statistically relevant conserved quantities for the truncated Burgers-Hopf equation
The Liouville property

Definition 7.1 (Liouville property) A vector field $\vec{F}(\vec{X})$ is said to satisfy the Liouville property if it is divergence free, i.e.

$$
div \vec{F} = \sum_{j=1}^{N} \frac{\partial F_j}{\partial X_j} = 0.
$$

(7.2)

An important consequence of the Liouville property is that it implies the flow map associated with (7.1) is volume preserving or measure preserving on the phase space. For this purpose we define the flow map $\{\Phi^t(\vec{X})\}_{t \geq 0}$, $\Phi^t : \mathcal{R}^N \mapsto \mathcal{R}^N$ associated with the finite-dimensional ODE system (7.1) by

$$
\frac{d}{dt} \Phi^t(\vec{X}) = \vec{F}(\Phi^t(\vec{X})), \quad \Phi^t(\vec{X})|_{t=0} = \vec{X}_0.
$$

(7.3)

Proposition 7.1 $\Phi^t(\vec{X})$ is volume preserving or measure preserving on the phase space, i.e.

$$
J(t) \overset{def}{=} \det \left( \nabla \vec{X} \Phi^t(\vec{X}) \right) \equiv 1,
$$

(7.4)

for all time $t \geq 0$. 

\[ \text{Di Qi, and Andrew J. Majda (CIMS)} \]
\[ \text{Truncated Burgers-Hopf Equations} \]
\[ \text{Oct. 27, 2016} \]
The Liouville property

**Definition 7.1 (Liouville property)** A vector field $\vec{F}(\vec{X})$ is said to satisfy the Liouville property if it is divergence free, i.e.

$$div \vec{F} = \sum_{j=1}^{N} \frac{\partial F_j}{\partial X_j} = 0. \quad (7.2)$$

An important consequence of the Liouville property is that it implies the flow map associated with (7.1) is volume preserving or measure preserving on the phase space. For this purpose we define the flow map $\{\Phi^t(\vec{X})\}_{t \geq 0}$, $\Phi^t : \mathcal{R}^N \mapsto \mathcal{R}^N$ associated with the finite-dimensional ODE system (7.1) by

$$\frac{d}{dt} \Phi^t(\vec{X}) = \vec{F}(\Phi^t(\vec{X})), \quad \Phi^t(\vec{X})|_{t=0} = \vec{X}_0. \quad (7.3)$$

**Proposition 7.1** $\Phi^t(\vec{X})$ is volume preserving or measure preserving on the phase space, i.e.

$$J(t)^{\text{def}} = \det \left( \nabla \Phi^t(\vec{X}) \right) \equiv 1, \quad (7.4)$$

for all time $t \geq 0$. 

$J(t)$ is the Jacobian determinant of the flow map $\Phi^t$. In other words, $J(t)$ represents the volume of the mapping $\Phi^t$.
Evolution of probability measures

Define a probability density \( p \left( \vec{X}, t \right) \) by the pull-back of the initial probability density \( p_0 \left( \vec{X} \right) \)

\[
p \left( \vec{X}, t \right) = p_0 \left( \left( \Phi^t \right)^{-1} \left( \vec{X} \right) \right),
\]

**Proposition 7.2** \( p(\vec{X}, t) = p_0 \left( \left( \Phi^t \right)^{-1}(\vec{X}) \right) \) is transported by the vector field \( \vec{F} \), i.e. it satisfies the Liouville’s equation

\[
\frac{\partial p}{\partial t} + \vec{F} \cdot \nabla_{\vec{X}} p = 0,
\]

(7.7)

and hence \( p(\vec{X}, t) \) is a probability density function for all time.

**Corollary 7.1** Let \( G(p) \) be any function of the probability density \( p \). Then

\[
\frac{\partial G(p)}{\partial t} + \vec{F} \cdot \nabla_{\vec{X}} G(p) = 0,
\]

(7.13)

i.e. \( G(p) \) satisfies Liouville’s equation. This further implies that

\[
\frac{d}{dt} \int_{\mathcal{R}^N} G(p(\vec{X}, t)) = 0.
\]

(7.14)
**Proof:** From the transport theorem (cf. Majda and Bertozzi, 2001 or Chorin and Marsden, 1993), we know that for any function \( f(\vec{X}, t) \) in phase space

\[
\frac{\partial}{\partial t} \int_{\Phi^t(\Omega)} f(\vec{X}, t) d\vec{X} = \int_{\Phi^t(\Omega)} \left( \frac{\partial f}{\partial t} + \text{div}(f \vec{F}) \right) d\vec{X},
\]

(7.8)

and since \( \vec{F}(\vec{X}, t) \) is divergence free, this equation reduces to

\[
\frac{\partial}{\partial t} \int_{\Phi^t(\Omega)} f(\vec{X}, t) d\vec{X} = \int_{\Phi^t(\Omega)} \left( \frac{\partial f}{\partial t} + \vec{F} \cdot \nabla_{\vec{X}} f \right) d\vec{X}.
\]

(7.9)

Next we let \( f(\vec{X}, t) = p(\vec{X}, t) \) in the last equation. From Definition 7.6 on the pull-back probability density function \( p \), we have

\[
\int_{\Phi^t(\Omega)} p(\vec{X}, t) d\vec{X} = \int_{\Phi^t(\Omega)} p_0 \left( (\Phi^t)^{-1}(\vec{X}) \right) d\vec{X}
\]

\[
= \int_{\Omega} p_0(\vec{Y}) \det \left( \nabla_{\vec{Y}} \Phi^t(\vec{Y}) \right) d\vec{Y} = \int_{\Omega} p_0(\vec{Y}) d\vec{Y},
\]

(7.10)
Conserved quantities

We now introduce the second ingredient for doing equilibrium statistical mechanics for ODEs. We assume that the system (7.1) possesses \( L \) conserved quantities \( E_l(\vec{X}(t)) \), i.e.

\[
E_l(\vec{X}(t)) = E_l(\vec{X}_0), \quad 1 \leq l \leq L. \tag{7.17}
\]

- For the truncated quasi-geostrophic equations: truncated energy, enstrophy, or higher moments;
- For the truncated Burgers-Hopf equation: the truncated energy, linear momentum, or Hamiltonian.

\[
\bar{E}_l = \langle E_l \rangle_p \equiv \int_{\mathbb{R}^N} E_l(\vec{X}) p(\vec{X}) d\vec{X}, \quad 1 \leq l \leq L. \tag{7.18}
\]

We naturally expect that these ensemble averages are conserved in time. Indeed, we have the following result:

**Proposition 7.3**

\[
\langle E_l \rangle_{p(t)} = \langle E_l \rangle_{p_0}, \text{ for all } t. \tag{7.19}
\]
Maximum entropy principle

The **Shannon entropy** \( S \) for the probability density function \( p(\vec{X}) \) on \( \mathbb{R}^N \) is absolutely continuous with respect to the Lebesgue measure by

\[
S(p) = -\int_{\mathbb{R}^N} p(\vec{X}) \ln p(\vec{X}) d\vec{X}.
\] (7.20)

Note that the Shannon entropy is exactly identical with the Boltzmann entropy in the statistical mechanics of gas particles. Furthermore, by setting \( G(p) = -p \ln p \) in Corollary 7.1, we can see that the Shannon entropy \( S \) is conserved in time

\[
S(p(t)) = S(p_0).
\] (7.21)

**maximum entropy principle**

\[
S(p^*) = \max_{p \in \mathcal{C}} S(p),
\]

subject to the constraint of measurements

\[
\mathcal{C} = \left\{ p(\vec{X}) \geq 0, \int_{\mathbb{R}^N} p(\vec{X}) d\vec{X} = 1, \langle E_l \rangle = \overline{E}_l, 1 \leq l \leq L \right\}.
\]
**Gibbs measure in the most probable state**

Variational derivative with respect to the probability density function

\[ \frac{\delta S}{\delta p} = -(1 + \ln p), \quad \frac{\delta E_l}{\delta p} = E_l (\vec{X}). \]

\[ p^* (\vec{X}) = c \exp \left( - \sum_{l=1}^{L} \theta_l E_l (\vec{X}) \right), \]

or equivalently, we may write the most probable probability density function in the form analogous to the **Gibbs measure** in statistical mechanics for gas particle systems

\[ p^* (\vec{X}) = \mathcal{G}_{\vec{\theta}} (\vec{X}) = C^{-1} \exp \left( - \sum_{l=1}^{L} \theta_l E_l (\vec{X}) \right), \quad (7.27) \]

provided that the constraints are normalizable, i.e.

\[ C = \int_{\mathcal{R}^N} \exp \left( - \sum_{l=1}^{L} \theta_l E_l (\vec{X}) \right) d\vec{X} < \infty \]

\[ (7.28) \]

and the \( \theta_l \)'s are the Lagrange multipliers so that \( \mathcal{G}_{\vec{\theta}} \) satisfies (7.18).
**Proposition 7.4** Let $E_j$, $1 \leq j \leq J$ be conserved quantities of the ODE (7.1). For any smooth function $G(E_1, \ldots, E_J)$, $G(E_1, \ldots, E_J)$ is a steady state solution to the Liouville equation. In particular, the most probable probability density function given by (7.27) is a steady state solution to the Liouville equation, i.e.

$$\tilde{F} \cdot \nabla_{\tilde{X}} \mathcal{G}_{\tilde{\theta}} = 0. \quad (7.29)$$

**Definition 7.2 (Invariant measure)** A measure $\mu$ on $\mathcal{R}^N$ is said to be invariant under the flow map $\Phi^t$ if

$$\mu((\Phi^t)^{-1}(\Omega)) = \mu(\Omega) \quad (7.31)$$

for all (measurable) set $\Omega \subset \mathcal{R}^N$ and all time $t$.

An easy consequence of this definition and proposition 7.4 is

**Corollary 7.2** The Gibbs measure $\mathcal{G}_{\tilde{\theta}}$ is an invariant measure of the dynamical system (7.1), i.e. it is a stationary statistical solution to (7.1).
Ergodicity and time averaging

- Taking the ensemble average of exact solutions analytically is an extremely challenging problem;
- Direct numerical approach to simulate a very large ensemble of trajectories is very costly in realistic systems;
- Instead, we usually assume that the Gibbs measure is ergodic so that the spatial average is equivalent to the time average:

\[ \langle g(\tilde{X}) \rangle = \int_{\mathbb{R}^N} g(\tilde{X}) \mathcal{G}_\theta(\tilde{X}) d\tilde{X} = \lim_{T \to \infty} \frac{1}{T} \int_{T_0}^{T_0+T} g(\tilde{X}(t)) \, dt \quad (7.32) \]

for almost all trajectories \( \tilde{X}(t) \). Such an ergodicity assumption allows us to verify the statistical prediction utilizing the long time average of one “typical” trajectory.
A simple example violating the Liouville property

The Liouville property for the vector field $\vec{F}$ is central for the above theory.

Let $\vec{X} = (X_1, X_2)^T$ and $\vec{F}^T = X_1 \vec{X}^\perp = (-X_1 X_2, X_1^2)$, i.e. consider the following dynamical system

$$\frac{dX_1}{dt} = -X_1 X_2, \quad \frac{dX_2}{dt} = X_1^2.$$ 

The system has the family of conserved quantities

$$E(\vec{X} \cdot \vec{X})$$

for an arbitrary smooth function, $E(s)$. Also, the vector field $\vec{F}$ violates the Liouville property since

$$\text{div}_\vec{X} \vec{F} = -X_2 \neq 0.$$
Statistical solutions of this equation satisfy the Fokker–Planck equation

\[
\frac{\partial p_t}{\partial t} + \text{div}_{\vec{X}}(\vec{F} p_t) = 0,
\]

and invariant measures like the Gibbs measure, which are functions of the conserved quantities, necessarily are steady solutions of this equations, i.e.

\[
\text{div}_{\vec{X}}(\vec{F} p_t) = 0.
\]

Now it is straightforward to verify that for any smooth conserved quantity \( E(\vec{X} \cdot \vec{X}) \)

\[
\text{div}_{\vec{X}}(\vec{F} E) \neq 0
\]

so that none of the non-trivial functions of the conserved quantities defines an invariant measure.

All solutions converge to the vertical axis \((X_1 = 0)\), and all invariant measures must be supported on the vertical axis

\[
\delta(X_1) \times \rho(X_2), \quad \text{for any } \rho \in PM(\mathcal{R})
\]

and thus it solves the Fokker–Planck equation in the sense of distribution and is not absolutely continuous with respect to the Lebesgue measure.
Statistical solutions of this equation satisfy the Fokker–Planck equation

$$\frac{\partial p_t}{\partial t} + \text{div}_X(\vec{F} p_t) = 0,$$

and invariant measures like the Gibbs measure, which are functions of the conserved quantities, necessarily are steady solutions of this equations, i.e.

$$\text{div}_X(\vec{F} p_t) = 0.$$

Now it is straightforward to verify that for any smooth conserved quantity $E(\vec{X} \cdot \vec{X})$

$$\text{div}_X(\vec{F} E) \neq 0$$

so that none of the non-trivial functions of the conserved quantities defines an invariant measure.

All solutions converge to the vertical axis ($X_1 = 0$), and all invariant measures must be supported on the vertical axis

$$\delta(X_1) \times \rho(X_2), \quad \text{for any } \rho \in PM(\mathcal{R})$$

and thus it solves the Fokker–Planck equation in the sense of distribution and is not absolutely continuous with respect to the Lebesgue measure.
Outline

1 Review: Introduction to statistical mechanics for ODEs

2 Statistical mechanics for the truncated Burgers-Hopf equations
   - The truncated Burgers-Hopf system
   - The Gibbs measure and the prediction of equipartition of energy
   - Numerical evidence of the validity of the statistical theory
   - TBH as a model with statistical features with atmosphere

3 Statistically relevant conserved quantities for the truncated Burgers-Hopf equation
The inviscid Burgers-Hopf equation

Since the atmosphere/ocean dynamics is extremely complicated, it makes sense to study simplified models that capture some specific features of the atmosphere/ocean dynamics.

The inviscid Burgers-Hopf equation

\[
\frac{\partial u}{\partial t} + \frac{1}{2} (u^2)_x = 0,
\]

is one of the extensively studied models in applied mathematics. One of the prominent features of this equation is the formation and propagation of shocks (discontinuity) from smooth initial data.

- Our interest here is models that capture certain salient features of the atmosphere/ocean dynamics;
- Consider spectral truncations of the Burgers-Hopf equation which conserves energy;
- Such approximation would be disastrous since shock waves form and dissipate energy.
The truncated Burgers-Hopf systems

Suppose the Burgers-Hopf equation is equipped with periodic boundary condition

\[
(u_\Lambda)_t + \frac{1}{2} P_\Lambda (u_\Lambda^2)_x = 0, \quad u_\Lambda \equiv P_\Lambda u = \sum_{|k| \leq \Lambda} \hat{u}_k(t) e^{ikx}.
\]

The equation can be written equivalently for the amplitudes \( \hat{u}_k \) with \( |k| \leq \Lambda \)

\[
\frac{d}{dt} \hat{u}_k = -\frac{ik}{2} \sum_{k=p+q, |p|, |q| \leq \Lambda} \hat{u}_p \hat{u}_q.
\]
Conservation of momentum and energy

It is elementary to show that solutions of the equations in either (7.35) or (7.37) conserve both momentum and energy, i.e.

\[ M = \int u(\Lambda)(t) = \hat{u}_0 \] (7.38)

and

\[ E = \frac{1}{2} \int u^2(\Lambda)dx = \frac{1}{2}|\hat{u}_0|^2 + \sum_{k=1}^{\Lambda} |\hat{u}_k|^2 \] (7.39)

are constant in time for solutions of (7.35) or (7.37). The proof for conservation of energy is as follows

\[ \frac{\partial}{\partial t} \int u^2(\Lambda) = -\frac{1}{2} \int u(\Lambda)P(\Lambda)(u^2)_x = -\frac{1}{2} \int u(\Lambda)(u^2)_x = -\frac{1}{3} \int (u^3)_x = 0. \]
Conservation of Hamiltonian

The momentum constraint in (7.38) is associated with trivial dynamical behavior and thus after a Galilean transformation, we have $M = 0$, so that $\hat{u}_0(t) \equiv 0$ in the formula for the energy, $E$, in (7.39). Also, all of the sums in (7.37) involve only $k$ with $1 \leq k \leq \Lambda$. In addition to the conserved quantities in (7.38) and (7.39), it is easy to check that the third moment

$$H = \int u_\Lambda^3 \, dx$$

(7.40)

is also conserved. Indeed

$$\frac{d}{dt} \int u_\Lambda^3 \, dx = 3 \int u_\Lambda^2 (u_\Lambda)_t \, dx = -\frac{3}{2} \int u_\Lambda^2 P_\Lambda (u_\Lambda^2)_x$$

$$= -\frac{3}{2} \int P_\Lambda (u_\Lambda^2) P_\Lambda (u_\Lambda^2)_x = 0.$$

The quantity, $H$, is actually the Hamiltonian for the truncated Burgers–Hopf system (see Abramov et al., 2003).
Checking the Liouville property

Let \( \hat{u}_k, 1 \leq k \leq \Lambda \) be the defining modes

\[
\hat{u}_k = a_k + ib_k, \quad 1 \leq k \leq \Lambda
\]  

(7.41)

with \( a_k \) and \( b_k \) being the real and imaginary part of \( \hat{u}_k \) respectively. Thus we have \( N = 2\Lambda \) number of unknowns. Let

\[
\vec{X} = (a_1, b_1, \ldots, a_{\Lambda}, b_{\Lambda}), \quad X_{2k-1} = a_k, \ X_{2k} = b_k, \quad 1 \leq k \leq \Lambda.
\]  

(7.42)

Equation (7.37) can then be written in a compact form

\[
\frac{d\vec{X}}{dt} = \vec{F}(\vec{X}), \quad \vec{X}(0) = \vec{X}_0.
\]

with the vector field \( \vec{F} \) satisfying the property

\[
\begin{align*}
F_{2k-1}(\vec{X}) &= k(X_{2k}X_{4k-1} - X_{2k-1}X_{4k}) \\
&\quad + \tilde{F}_{2k-1}(X_1, \ldots, X_{2k-2}, X_{2k+1}, \ldots, X_N) \\
F_{2k}(\vec{X}) &= k(X_{2k-1}X_{4k-1} + X_{2k}X_{4k}) \\
&\quad + \tilde{F}_{2k}(X_1, \ldots, X_{2k-2}, X_{2k+1}, \ldots, X_N),
\end{align*}
\]  

(7.43)
The Gibbs measure from the energy

Assume that the momentum, \( M_0 = 0 \), the most probable distribution, or the canonical Gibbs measure is given by

\[
\mathcal{G}_\theta = C_\theta^{-1} \exp \left( -\frac{\theta}{2} \sum_{k=1}^{\Lambda} |\hat{u}_k|^2 \right), \quad \theta > 0. \tag{7.44}
\]

This is a product of identical Gaussian measures. For a given value of the mean energy \( \bar{E} \), the Lagrange multiplier \( \theta \) is given by, thanks to (7.39)

\[
\theta = \frac{\Lambda}{\bar{E}}, \quad \text{and} \quad \text{Var}\{a_k\} = \text{Var}\{b_k\} = \frac{1}{\theta} = \frac{\bar{E}}{\Lambda} \tag{7.45}
\]

where \( \text{Var} \) denotes the variance. The canonical Gibbs ensemble predicts a zero mean state and a spectrum with *equi-partition of energy* in all modes according to (7.44).
Next we ask whether the same procedure would work for the conserved quantity given by \( H = \int u^3 \Lambda dx \) from (7.40). This is an odd function of \( u_\Lambda \) and grows cubically at infinity so that the associated Gibbs measure is infinite. Thus, this conserved quantity provides an example which is not statistically normalizable so that the integral in (7.28) is infinite even though it is the Hamiltonian for this system. We discuss the statistical relevance of this conserved quantity for the dynamics briefly later in this book (see Abramov et al., 2003).
Numerical set-up for the truncated Burgers-Hopf equation

We show the statistical prediction of equi-partition of energy in all modes for $\Lambda$ of moderate size.

- A pseudo-spectral method of spatial integration is used;
- Fourth-order Runge-Kutta time stepping is utilized with typical time step $\Delta t = 2 \times 10^{-4}$;
- The initial averaging value, $T_0 = 100$, and averaging window, $T = 5000$, are utilized.

All statistical quantities are computed as time averages, i.e. we assume that the Gibbs measure (7.44) is ergodic and hence (7.32) applies. In particular, the energy in the $k$th mode is computed by

$$\langle |\hat{u}_k|^2 \rangle = \frac{1}{T} \int_{T_0}^{T+T_0} |\hat{u}_k(t)|^2 dt. \quad (7.46)$$
Numerical set-up for the truncated Burgers-Hopf equation

Here we discuss the case with $\theta = 10, \Lambda = 100$.

The initial data are selected at random with Fourier coefficients $u_k$, $1 \leq k < \Lambda - 15$ sampled from a Gaussian distribution with mean zero and variance $0.8 \times \theta^{-1}$. The tail Fourier coefficients with $\Lambda - 15 \leq k \leq \Lambda$ are initialized with random phases and equal amplitudes

$$|u_k|^2 = \frac{1}{15} \left( E - \sum_{j=1}^{\Lambda-16} |u_j|^2 \right), \quad \Lambda - 15 \leq k \leq \Lambda$$

to satisfy the energy constraint given by $\overline{E}$. 

A wide range of simulations of (7.37) have been performed in several parameter regimes. Here we discuss the case $\theta = 10$ and $\Lambda = 100$. Truncations varying both $\Lambda$ and $\theta$ (Majda and Timofeyev, 2000, 2002) exhibit qualitatively similar behavior and we focus our attention on the numerical simulations with $\theta = 10, \Lambda = 100$. Many more results of this type can be found in the two papers mentioned above.

The energy spectrum for the real and imaginary parts of the Fourier modes is presented in the top and the bottom parts of Figure 7.1, respectively. The straight lines in Figure 7.1 correspond to the theoretically predicted value $\overline{E}$. 

Figure 7.1 Galerkin truncation with $\theta = 10, \Lambda = 100$. Energy spectrum; circles – Numerics, solid line – analytical predictions. Note the small vertical scale consistent with errors of at most a few percent.
Energy spectrum for the real and imaginary parts of the Fourier modes

Figure 7.1 Galerkin truncation with $\theta = 10$, $\Lambda = 100$. Energy spectrum; circles – Numerics, solid line – analytical predictions. Note the small vertical scale consistent with errors of at most a few percent.
To check the Gaussianity of the dynamics we compute the relative error between the analytical predictions in (7.47) and numerical estimates given by

\[
\text{Rel. Err.\{4th Moment\}} = |\langle|\text{Re} \hat{u}_k|^4\rangle - \frac{3}{4} \theta^{-2}| \times \frac{4}{3} \theta^2
\]

\[
\text{Rel. Err.\{6th Moment\}} = |\langle|\text{Re} \hat{u}_k|^6\rangle - \frac{15}{8} \theta^{-3}| \times \frac{8}{15} \theta^3
\]

\[
\text{Rel. Err.\{8th Moment\}} = |\langle|\text{Re} \hat{u}_k|^8\rangle - \frac{105}{16} \theta^{-4}| \times \frac{16}{105} \theta^4.
\]
TBH as a model with statistical features with atmosphere

The truncated Burgers-Hopf equation is a simple model with many degrees of freedom with many statistical properties similar to those occurring in dynamical systems relevant to the atmosphere.

- Long time correlated large-scale modes for low frequency variability;
- Short time correlated “weather modes” at small scales;
- A simple correlation scaling theory for the modes is developed.
A scaling theory for temporal correlations

It is a simple matter to present a scaling theory which predicts that the temporal correlation times of the large-scale modes are longer than those for the small-scale modes. Recall from (7.45) that the statistical predictions for the energy per mode is \( \bar{E}/\Lambda = \theta^{-1} \); since \( \bar{E}/\Lambda \) has units length\(^2\)/time\(^2\), and the wave number \( k \) has units length\(^{-1}\), the predicted eddy turnover time for the \( k \)th mode is given by

\[
\left( \frac{\Lambda}{\bar{E}} \right)^{1/2} \frac{1}{k} = \frac{\sqrt{\theta}}{k}, \quad 1 \leq k \leq \Lambda.
\]

If the physical assumption is made that the \( k \)th mode decorrelates on a time scale proportional to the eddy turnover time with a universal constant of proportionality, \( C_0 \), a simple plausible scaling theory for the dynamics of the equation in (7.35) or (7.37) emerges and predicts that the correlation time for the \( k \)th mode, \( T_k \), is given by

\[
T_k = \frac{C_0 \sqrt{\theta}}{k}, \quad 1 \leq k \leq \Lambda. \quad (7.49)
\]
Numerical evidence for the correlation scaling theory

the correlation function $C_k(\tau)$ is computed by

$$
C_k(\tau) = \frac{1}{T} \int_{T_0}^{T_0+T} X_k(t+\tau)X_k(t)dt,
$$

(7.50)

with $T$ picked sufficiently large. We also define the correlation time for the $k$th mode through

$$
T_k = \int_0^{\infty} |C_k(\tau)|d\tau.
$$

(7.51)

Here results are presented for the same case $\theta = 10, \Lambda = 100$

(a) Correlation functions for $k = 1, 2, 3, 10, 15, 20$

(b) Correlation time v. wavenumber $k$. DNS & scaling
Outline

1 Review: Introduction to statistical mechanics for ODEs

2 Statistical mechanics for the truncated Burgers-Hopf equations
   - The truncated Burgers-Hopf system
   - The Gibbs measure and the prediction of equipartition of energy
   - Numerical evidence of the validity of the statistical theory
   - TBH as a model with statistical features with atmosphere

3 Statistically relevant conserved quantities for the truncated Burgers-Hopf equation
The TBH equations

\[(u^\Lambda)_t + \frac{1}{2} P^\Lambda (u^\Lambda_2) = 0, \quad |k| \leq \Lambda,\]

have three conserved quantities, the momentum \(M\), energy \(E\), and Hamiltonian \(H\)

\[
M = \int_{\Omega} u^\Lambda, \quad E = \frac{1}{2} \int_{\Omega} P^\Lambda (u^\Lambda_2), \quad H = \int_{\Omega} P^\Lambda (u^\Lambda_3).
\]

- We showed a non-trivial Gibbs measure with equipartition of energy associated the energy \(E\) alone;
- The question arises regarding the statistical significance of the conserved quantity \(H^2\).
For statistically relevant values of the Hamiltonian, the direct numerical simulations showed a surprising spectral tilt rather than equipartition of energy.

This spectral tilt was predicted and confirmed independently by Monte-Carlo simulations based on equilibrium statistical mechanics together with a heuristic formula for the spectral tilt,

$$\frac{1}{2} \left\langle |\hat{u}_k|^2 \right\rangle = \frac{E}{\Lambda} \left( 1 + \frac{8}{5} \frac{H^2}{E^3} \frac{(\Lambda + 1)/2 - k}{\Lambda} \right),$$

where $\Lambda$ denotes the number of degrees of freedom, $E$ is energy, and $H$ is Hamiltonian.
The linear spectral tilt (Strong Mixing)

The energy spectrum for the simulation with $\Lambda = 100, H = 0.01$. Solid line – linear fit found by least squares, dash-dotted line – linear fit (78), dashed line shows equipartition

**FIG. 15.** The spectral tilt and the time correlation function for the case $\Lambda = 100, H = 0.01$. The correlation function confirms mixing, the spectral tilt is linear, and the predicted tilt coincides with the actual slope
The linear spectral tilt (Weak Mixing)

The energy spectrum for the simulation with $\Lambda = 100$, $H = 0.15$. Solid line – linear fit found by least squares, dash-dotted line – linear fit (78), dashed line shows equipartition

Fig. 19. The spectral tilt and the time correlation function for the case $\Lambda = 100$, $H = 0.15$. Long correlations show failure of mixing, the spectral tilt is nonlinear and the predicted tilt is much more steeper than the actual slope.

The time correlation function for the Fourier mode $k = 1$, $\Lambda = 100$, $H = 0.15$.
Questions & Discussions