1. Set-up, notations

- $T_\lambda$ – Galton-Watson tree with Poisson($\lambda$) offspring distribution. We consider $\lambda > 1$.
- $\mathcal{T}$ space of all rooted trees with finite-degree vertices.
- For $T \in \mathcal{T}$ and $n \in \mathbb{N}$, $T|_n$ is the truncation of $T$, where everything below generation $n$ have been removed (the root $\rho$ is at generation 0).
- For $v \in V(T)$, $T_v$ denotes the subtree coming out of $v$ in $T$. For $v \in V(T_\lambda)$, let $T_{\lambda,v}$ denote the random subtree rooted at $v$ in $T_\lambda$.
- For $n \in \mathbb{N}$, let $L_n(T)$ denote the set of all nodes at generation $n$ of $T$, and let $L_n(T_\lambda)$ be denoted in short by $L_n$.

Consider property $A = \{ \exists$ a complete binary tree as a subtree, starting at $\rho \}$. Want to find $P[T_\lambda \text{ satisfies } A] = p_\lambda(A)$. If $T_\lambda$ satisfies $A$, then the root $\rho$ must have at least two children $u,v$ such that $T_{\lambda,u}$ and $T_{\lambda,v}$ also satisfy property $A$. If we colour the root of every tree that satisfies $A$ by 1, and the root of every other tree by 0, then we have the following rule:

$$u \in T \text{ is coloured } 1 \text{ iff it has at least two children } v_1, v_2 \text{ coloured } 1.$$  \hspace{1cm} (1.1)

If the probability of a node being coloured 1 under the Galton-Watson regime is $x$, then using self-similarity of the GW process and Poisson thinning, the number of children of a node that are coloured 1 follows Poisson($\lambda x$). Hence $x$ satisfies

$$x = P[\text{Poi}(\lambda x) \geq 2] = 1 - e^{-\lambda x}(1 + \lambda x) = \Psi_\lambda(x).$$  \hspace{1cm} (1.2)

Thus, $p_\lambda(A)$ will be a fixed point of $\Psi_\lambda$. But for all $\lambda$ bigger than a critical $\lambda_0 \approx 3.35$, $\Psi_\lambda$ has three fixed points: 0, the true solution and a third solution, as shown below:

The green solution is the true probability as a function of $\lambda$.

What does the 0 solution (purple) stand for? If you deliberately colour every node of every tree in $\mathcal{T}$ 0, then it still satisfies the rule $[1.1]$, but now the probability of being painted 1 is 0.

**Question:** What does the red, middle solution, stand for?
2. Formulating the Question

Of course, we have to formally state the question mentioned above. And we shall do it for more general scenarios.

**Definition 2.1.** A tree automaton (finite state tree automaton) $A$ consists of a finite state space $\Sigma$ of colours, a positive integer $k$ and a map (rule) $\Gamma : \{0, 1, \ldots, k\}^\Sigma \to \Sigma$.

**Definition 2.2.** For any tree $T \in \mathcal{T}$ and any assignment $\omega : V(T) \to \Sigma$, we call $\omega$ compatible with respect to $A$ on $T$ if for every $v \in V(T)$, we have $\omega (v) = \Gamma (\bar{n})$, where $\bar{n} = (n_{\sigma} \wedge k : \sigma \in \Sigma)$, where $n_{\sigma}$ is the number of children $u$ of $v$ with $\omega (u) = \sigma$.

**Definition 2.3.** Call a map $f : \mathcal{T} \to \Sigma$ an interpretation for $A$ if the assignment $\omega : V(T_\lambda) \to \Sigma$ defined as $\omega (v) = f (T_{\lambda, v})$ for all $v \in V(T_\lambda)$ is almost surely compatible.

Let $p_\lambda (\sigma)$ be the probability of being painted $\sigma \in \Sigma$. Then using Poisson thinning, we know that

$$p_\lambda (\sigma) = \sum_{\bar{n} = (n_{\gamma} ; \gamma \in \Sigma) \in \Sigma, \Gamma (n_{\gamma} \wedge k ; \gamma \in \Sigma) = \sigma} \prod_{\gamma \in \Sigma} P [\text{Poi}(\lambda p_\lambda (\gamma)) = n_{\gamma}]. \quad (2.1)$$

We can consider the set $D$ of all possible probability distributions on $\Sigma$. If we define the recursion $\Psi : D \to D$ such that for $\tilde{\gamma} \in D$, $\Psi (\tilde{\gamma}) = (\Psi_\sigma (\tilde{\gamma}) : \sigma \in \Sigma)$ with

$$\Psi_\sigma (\tilde{\gamma}) = \sum_{\bar{n} = (n_{\gamma} ; \gamma \in \Sigma) \in \Sigma, \Gamma (n_{\gamma} \wedge k ; \gamma \in \Sigma) = \sigma} \prod_{\gamma \in \Sigma} P [\text{Poi}(\lambda y_{\gamma}) = n_{\gamma}], \quad (2.2)$$

then $\tilde{\mu}_\lambda = (p_\lambda (\sigma) : \sigma \in \Sigma)$ is a fixed point of $\Psi$.

**Observation 2.4.** $f(T_\lambda)$ is a fixed point of $\Psi$.

If for a particular fixed point $\nu$ of $\Psi$, we have $f(T_\lambda) = \nu$, then we say that $f$ is an interpretation corresponding to $\nu$ for the automaton $A$. Note that, we do not want $f$ to be a random assignment. We call a fixed point $\nu$ rogue if no such interpretation corresponding to it exists.

We will only consider $\Sigma = \{0, 1\}$ for this talk.

3. Condition for Existence of Interpretation

**Lemma 3.1.** For any fixed point $\nu$ of $\Psi$, there exists a compatible assignment $\omega$ on $T_\lambda$ such that conditioned on $T_\lambda|_n$, we have $(\omega (v) : v \in L_n)$ i.i.d. $\nu$.

**Idea of proof:** For every $n \in \mathbb{N}$, assign i.i.d. colours 0 or 1 to the nodes in $L_n$ according to $\nu$, then use the rule $\Gamma$ of the automaton to get the colours of all the nodes in $T|_{n-1}$. Then either use compactness argument via a diagonal sequence, or the Kolmogorov extension theorem, to extend to the whole of infinite trees.

Then why does this assignment not work to give us an interpretation? Because we want a deterministic interpretation. This will not provide us that.

**Lemma 3.2.** For $\nu$ a fixed point of $\Psi$ and the $\omega$ mentioned in Lemma 3.1, if

$$\lim_{n \to \infty} P \left[ \omega (\rho) = \sigma | T_\lambda|_n \in \{0, 1\} \right] \text{ almost surely, for } \sigma \in \{0, 1\}, \quad (3.1)$$

then $\nu$ has a unique interpretation up to measure 0. If not, then $\nu$ is rogue.

**Idea of proof:** By Levy’s upward theorem, almost surely

$$\lim_{n \to \infty} P \left[ \omega (\rho) = \sigma | T_\lambda|_n \right] = P \left[ \omega (\rho) = \sigma | T_\lambda \right]. \quad (3.2)$$

If an interpretation $f$ does exist, then we know that given the realization $T$ of $T_\lambda$, $\omega (\rho) = f(T)$ will be completely determined, and hence the right hand side of (3.2) will be either 0 or 1, no other value in between. Hence (3.1) has to hold for an interpretation to exist.
3.1. Pivot tree.

**Definition 3.3.** Call \( v \in L_n(T) \) for a \( T \in \mathcal{T} \) and a compatible assignment \( \omega \) on \( T \) pivotal if changing the colour of \( \omega(v) \) to \( 1 - \omega(v) \), and keeping the colours of all other nodes in \( L_n \) unchanged, changes the colour of the root from \( \omega(\rho) \) to \( 1 - \omega(\rho) \), according to the rules \( \Gamma \) of the automaton.

Note that if \( v \) is pivotal and \( u \) is the parent of \( v \), then either \( u = \rho \), or, changing the colour of \( v \) also changes the colour of \( u \), since otherwise the change cannot affect the root’s colour. This means that the subgraph induced by only the pivotal nodes (where the root is trivially pivotal) is actually a subtree. Call this subtree \( T_{\text{piv}} \). For \( T_\lambda \), call it \( T_{\lambda, \text{piv}} \).

**Lemma 3.4.** For the \( \omega \) mentioned in Lemma 3.1 both \( (T_\lambda, \omega) \) and \( (T_{\lambda, \text{piv}}, \omega_{\text{piv}}) \) are multi-type Galton-Watson trees (where \( \omega_{\text{piv}} \) is \( \omega \) restricted to \( T_{\lambda, \text{piv}} \)).

**Idea of proof:** Can be proved more easily when the random colours on the nodes of the tree are generated simultaneously while the tree is being grown according the to the Galton-Watson process, using Poisson thinning conditioned properly.

For a given tree \( t \), and a given compatible assignment \( \omega \), we can find out exactly which nodes in \( L_n(t) \) pivotal. We can use the \( \omega \) we found in Lemma 3.1. Now, it makes perfect sense to define \( P_n \) to be the random number of pivotal nodes in \( L_n \) of the coloured tree \( (T_\lambda, \omega) \) for \( \omega \) as in Lemma 3.1. It also makes sense to define

\[
I_n(t) = E[|P_n|T_\lambda|n = t|n], \text{ for } t \in \mathcal{T}.
\]

Toby’s ingenious idea: to think of \( I_n(t) \) as an influence, and use a variation of the KKL inequality.

**Lemma 3.5.** The sequence \( \{I_n(T_\lambda) : n \in \mathbb{N}\} \), with respect to the filtration generated by \( T_\lambda|n \), is a supermartingale if \( T_{\lambda, \text{piv}} \) is subcritical, and submartingale if \( T_{\lambda, \text{piv}} \) is supercritical.

I deliberately steer clear of criticality in this short talk.

**Idea of proof:** Note that for every pivotal node \( v \) in \( L_{n-1} \), if \( \omega(v) = 1 \), then it will have a random number \( n_0 \) of pivotal children labeled 0 and a random number \( n_1 \) of pivotal children labeled 1, and this depends on the offspring distribution of \( T_{\lambda, \text{piv}} \) which is a multi-type Galton-Watson tree. Similarly, for every pivotal node \( v \) in \( L_{n-1} \), if \( \omega(v) = 0 \), then it will have a random number \( m_0 \) of pivotal children labeled 0 and a random number \( m_1 \) of pivotal children labeled 1, and this depends on the offspring distribution of \( T_{\lambda, \text{piv}} \) which is a multi-type Galton-Watson tree.

\[
E[I_n(T_\lambda)|T_\lambda|n-1] = E[P_n|T_\lambda|n] = E[P_nT_{\lambda,piv}|n-1], \text{ by tower property of conditional expectation;}
\]

\[
\approx \text{overall growth rate of } T_{\lambda,piv} \cdot P_{n-1}.
\]

where the last step is not quite accurate. You need to be more careful as this is a multi-type Galton-Watson tree you are dealing with (so you ought to look at the mean matrix and its largest eigenvalue etc.). Clearly, of the overall growth rate of \( T_{\lambda,piv} \) is \( < 1 \), then we indeed get a supermartingale.

**Lemma 3.6.** If \( T_{\lambda,piv} \) is subcritical, then \( I_n(T_\lambda) \) converges almost surely, and in this case actually the limit is 0.

This follows because from Lemma 3.5 we now know that \( I_n(T_\lambda) \) is a nonnegative supermartingale.

**Lemma 3.7.** If \( T_{\lambda,piv} \) is subcritical, then the associated fixed point \( \nu \) is interpretable, i.e. not rogue.

**Very rough idea of the proof:** Essentially have to show that \( \{3.1\} \) holds, i.e.

\[
\lim_{n \to \infty} P[\omega(\rho) = \sigma|T_\lambda|n] \in \{0, 1\} \text{ almost surely, for } \sigma \in \{0, 1\}.
\]

For this, we utilize the fact that \( P[\omega(\rho) = \sigma|T_\lambda|n] \) can be bounded using a variation of the KKL inequality, and the influence function used therein can be further bounded above by \( I_n(T_\lambda) \) which we know goes to 0.