Almost sure theory for first order logic on Galton-Watson trees and probabilities of local neighbourhoods of the root

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Joint work with Joel Spencer

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The First Order (F.O.) World

A random Galton-Watson tree, $Poisson(\lambda)$ offspring distribution.
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The First Order (F.O.) World

$T$ random Galton-Watson tree, Poisson($\lambda$) offspring distribution.

- Constant Symbol: root
- Equality: $x = y$,
- Parent: $\pi(y) = x$ ($x$ is parent of $y$, binary predicate),
- Variable Symbols $x, y, z \ldots$,
- Boolean $\lor, \land, \neg, \rightarrow, \leftrightarrow$, etc,
- Quantification $\forall x, \exists y$ over vertices only.
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Example

$\exists$ a node with exactly one child and one grandchild.
Ehrenfeucht games

Definition

1. Trees $T_1, T_2$, roots $R_1, R_2$, $\#$ moves $= k$. 
Ehrenfeucht games

Definition

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2. Spoiler picks any one tree and a node from it. Duplicator chooses a node from the other tree.

Theorem
If Duplicator wins $EHR[T_1, T_2, k]$ then $T_1 \models \varphi$ $\iff$ $T_2 \models \varphi$ for $\text{FO}_A$ of depth $k$. 
Ehrenfeucht games

Definition

1. Trees $T_1$, $T_2$, roots $R_1$, $R_2$, $\# \text{ moves } = k$.
2. Spoiler picks any one tree and a node from it. Duplicator chooses a node from the other tree.
3. $(x_i, y_i) \in T_1 \times T_2, 1 \leq i \leq k$, pairs of nodes selected.
### Ehrenfeucht games

#### Definition

1. **Trees** $T_1, T_2$, roots $R_1, R_2$, \# moves $= k$.
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3. $(x_i, y_i) \in T_1 \times T_2, 1 \leq i \leq k$, pairs of nodes selected.
4. **Duplicator wins if**
   - $x_i = R_1 \iff y_i = R_2$,
Ehrenfeucht games

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   - $\pi(x_j) = x_i \iff \pi(y_j) = y_i$, for $\pi$. 

Theorem: If Duplicator wins $EHR\left[ T_1, T_2, k \right]$ then $T_1 | = A \iff T_2 | = A$ for $F.O.A$ of depth $k$. 
Ehrenfeucht games

**Definition**

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Theorem

If Duplicator wins $EHR[T_1, T_2, k]$ then $T_1 | A \iff T_2 | A$ for F. O. A of depth $k$. 

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Ehrenfeucht games

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Theorem

If Duplicator wins $EHR[T_1, T_2, k]$ then

$$T_1 \models A \iff T_2 \models A$$

for F.O. $A$ of depth $k$. 
Ehrenfeucht value

Definition

$T_1 \equiv_k T_2$ if Duplicator wins $EHR[T_1, T_2, k]$. 
Ehrenfeucht value

**Definition**

\[ T_1 \equiv_k T_2 \text{ if Duplicator wins } EHR[T_1, T_2, k]. \]

**Theorem**

*Fix k. Only finitely many equivalence classes.*
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Ehrenfeucht value

**Definition**

\[ T_1 \equiv_k T_2 \] if Duplicator wins \( \text{EHR}[T_1, T_2, k] \).

**Theorem**

Fix \( k \). Only finitely many equivalence classes.

**Definition**

Equivalence class of \( T \) its Ehrenfeucht value.
Our results on almost sure theory for F.O.

**Theorem**

- Fix $k \in \mathbb{N}$.

- Fix a finite tree $T_0$. $A[T_0] := \{ \exists \text{ a subtree } \cong T_0 \text{ in } T \}$.

- Conditioned on the tree being infinite, $A$ is almost surely true.

- Schema $A = \{ A[T_0] : \forall \ T_0 \text{ finite tree} \}$ gives almost sure theory for infinite trees.
Consequence of previous result

**Corollary**

- **Fix** $k \in \mathbb{N}$. **Condition on** $T$ **being infinite**.

- **Ehrenfeucht value of** $T$ **depends on the local neighbourhood of the root, of radius** $\approx 3^{k+2}$.

- **For all** $A = A[T_0]$, $P[A] = P[A^*]$ **where** $A^*$ **only depends on the local neighbourhood of the root**.
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First generation probability conditioned on infiniteness

1. Only concerned with $\Gamma_1 = \{0, 1, 2, \ldots k - 1, \omega\}$, $\omega$ indicates $\geq k$. 

\[ P[ A_\omega \cap B^c ] = P[A_\omega] - P[A_\omega \cap B] = \sum_{j=k}^{\infty} e^{-\lambda} \cdot \lambda^j j! \left[ 1 - p_j \right]. \]
First generation probability conditioned on infiniteness

1. Only concerned with $\Gamma_1 = \{0, 1, 2, \ldots k - 1, \omega\}$, $\omega$ indicates $\geq k$.
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3. \( B = \{T \text{ is finite}\} \), \( P[B] = p \).

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4. For $i \in \{0, 1, \ldots k - 1\}$

$$P[A_i \cap B^c] = P[A_i] - P[A_i \cap B]$$

$$= e^{-\lambda} \cdot \frac{\lambda^i}{i!} (1 - p^i).$$
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\[
P[A_i \cap B^c] = P[A_i] - P[A_i \cap B] = e^{-\lambda} \cdot \frac{\lambda^i}{i!} (1 - p^i).
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5. For $\omega$ children:

\[
P[A_\omega \cap B^c] = P[A_\omega] - P[A_\omega \cap B] = \sum_{j=k}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} [1 - p^j].
\]
Some definitions

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Definition

1. For $0 \leq i \leq k - 1$, $P_i(x) = \Pr[Poi(x) = i] = e^{-x} x^i / i!$.

2. For $i \geq k$, $P_\omega(x) = \Pr[Poi(x) \geq k] = e^{-x} \sum_{j=k}^{\infty} x^j / j!$.

3. $(i+1)$-generation neighbourhood $\Gamma_{i+1} = \{ g : \Gamma_i \to \Gamma_1 \}$.

4. For $\tau \in \Gamma_i$, $P_\tau(x) = \Pr[i \text{-generation neighbourhood is in } \tau \text{ for } Poi(x)]$. 
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Example + Illustration

\[ k = 3, g(0) = 0, g(1) = \omega, g(2) = 1, g(\omega) = 1. \]

\[ P_{\sigma}(x) = 2 \prod_{i=0}^{x} P_{g(i)}(xP_{i}(x)) \cdot P_{g(\omega)}(xP_{\omega}(x)) = e^{-x} \cdot \left( \sum_{j=3}^{\infty} \left( xP_{1}(x) \right)^{j} \cdot j! \right) \cdot \left( xP_{2}(x) \right) \cdot \left( xP_{\omega}(x) \right) \]
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\[ P_{\sigma}(x) = \prod_{i=0}^{2} P_{g(i)}(xP_i(x)) \cdot P_{g(\omega)}(xP_\omega(x)) \]

\[ = e^{-x} \cdot \left\{ \sum_{j=3}^{\infty} \frac{(xP_1(x))^j}{j!} \right\} \cdot (xP_2(x)) \cdot (xP_\omega(x)) \]
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Recursive computation of probabilities

**Theorem**

\[ P_\sigma(x) = \prod_{\tau \in \Gamma_i} P_{g(\tau)}(xP_{\tau}(x)) \quad \forall \sigma = g \in \Gamma_{i+1}. \]
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Theorem

$$B = \{ T \text{ is finite} \}, \quad P[B] = p.$$
Recursive computation of probabilities

Theorem

\[ P_\sigma(x) = \prod_{\tau \in \Gamma_i} P_g(\tau)(xP_\tau(x)) \quad \forall \, \sigma = g \in \Gamma_{i+1}. \]

Theorem

- \( B = \{ T \text{ is finite} \} \), \( P[B] = p. \)
- By duality for Galton-Watson trees,

\[ P\{i - \text{generation neighbourhood} = \sigma\} | B] = P_\sigma(p\lambda). \]
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Theorem

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- By duality for Galton-Watson trees,
  \[ P[\{ \text{i\text{-}generation neighbourhood} = \sigma \}|B] = P_\sigma(p\lambda). \]

- If \( P^*[\sigma] = P[\{ \text{i\text{-}generation neighbourhood} = \sigma \}|B^c], \) then
  \[ P^*[\sigma] = \frac{P_\sigma(\lambda) - p \cdot P_\sigma(p\lambda)}{1 - p}. \]
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The probabilities are *nice* functions

**Remark**

*For all* $i$ *and* $\sigma \in \Gamma_i$, $P_\sigma(x)$ *nice function. Consists of polynomials in* $p$, $x$, $e^{-x}$, *and base* $e$ *exponentiation.*
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**Example**

1. $A := \{ \text{Root has no child with no child} \}, \ k \geq 1.$
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2. $A = \{ g(0) = 0 \}$.
3. $P[A] = P_0[\lambda P_0(\lambda)] = e^{-\lambda P_0(\lambda)} = e^{-\lambda}e^{-\lambda}$. 
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Thank you.