The Strange Logic of Galton-Watson Trees

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Galton-Watson branching process

- Root $R$ has Poisson($\lambda$) children.
- Each child of $R$ independently has Poisson($\lambda$) children. So on.
- Denote by $T_\lambda$. May be finite or infinite. Survives with positive probability for $\lambda > 1$.
- Our results extend to very general offspring distributions, as long as they are light tailed.
- Poisson thinning allows nicer computations.
First order world

Joint work with Prof. Joel Spencer.
First order (FO) properties

Captures finite, local structures inside the tree.
First order: the formal view

1. Constant Symbol: root;
2. Equality: \( x = y \);
3. Parent: \( \pi(y) = x \) (\( x \) is the parent of \( y \));
4. Variable Symbols \( x, y, z \ldots \), i.e. the nodes;
5. Boolean connectives \( \lor, \land, \neg, \Rightarrow, \Leftrightarrow \), etc;
6. Quantification \( \forall, \exists \), over vertices only.

Quantifier depth is \# nested quantifiers.
First order: examples and summary

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2. The root has no child with no child with no child.

Summary

1. Fix a finite tree $T_0$. Let $A[T_0] := \{T_\lambda \text{ contains a subtree } \sim = T_0\}$.
2. Specifying a neighbourhood of the root is also FO.
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3. The root has exactly 2 children, one of which is childless and the other has exactly 3 children.
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Summary

1. Fix a finite tree $T_0$. Let $A[T_0] := \{T : T \lambda$ contains a subtree $\cong T_0\}$. This is FO.
2. Specifying a neighbourhood of the root is also FO.
Equivalence classes

- Fix arbitrary $k$.

- Define the relation $\sim$: trees $T_1 \sim T_2$ if they satisfy the same FO properties of quantifier depth $\leq k$.

- $\sim$ is an equivalence relation. Partitions rooted trees into set $\Sigma = \Sigma_k$ of equivalence classes.

- There are only finitely many equivalence classes, i.e. $\Sigma$ finite.

- **Aim:** To obtain $P[T_\lambda \in \sigma] = P_\lambda(\sigma)$, for $\sigma \in \Sigma$.

- Given FO $A$ with quantifier depth $\leq k$, find all $\sigma \in \Sigma$, such that $T \models A$ for $T \in \sigma$. Then $P_\lambda[A] = \sum^* P_\lambda(\sigma)$. 
A recursion of the equivalence classes

For $\sigma \in \Sigma$, let $n_\sigma = \# \{ u_i : T(u_i) \in \sigma \}$.

The vector $\vec{n} = (\min\{n_\sigma, k\} : \sigma \in \Sigma)$ will determine equivalence class of the entire tree.
A recursion of the equivalence classes

▶ For $\sigma \in \Sigma$, let $n_\sigma = \#\{u_i : T(u_i) \in \sigma\}$.
▶ The vector $\vec{n} = \left( \min \{n_\sigma, k\} : \sigma \in \Sigma \right)$ will determine equivalence class of the entire tree.

![Diagram of a tree with recursion notation and equivalence classes]
A distributional map

- $\vec{n} = \left( \min \{n_\sigma, k\} : \sigma \in \Sigma \right)$ determines equivalence class of the entire tree.

- Exist deterministic rules $\Gamma$, such that $\Gamma(\vec{n})$ gives the equivalence class of the entire tree.

- $D$ set of all probability distributions on $\Sigma$. Define distributional map $\Psi : D \to D$ as follows.

- Choose $\vec{x} \in D$. Let root have $\text{Poi}(\lambda)$ children.

- Assign equivalence class, mutually independently, to each child of root, according to $\vec{x}$. Get induced equivalence class at the root, according to $\Gamma$.

- Let $\Psi(\vec{x})$ be distribution of the induced equivalence class at the root.
Contracting map and unique fixed point

Observation

If \( \vec{p}(\lambda) = (P_\lambda(\sigma) : \sigma \in \Sigma) \), then \( \vec{p}(\lambda) \) is a fixed point of \( \Psi \).

Remark

The proof techniques differ widely for \( \lambda < 1 \) and \( \lambda \geq 1 \).
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If $\vec{p}(\lambda) = (P_\lambda(\sigma) : \sigma \in \Sigma)$, then $\vec{p}(\lambda)$ fixed point of $\Psi$.

Theorem (P., Spencer)

1. $\Psi$ is a contraction.

2. $\vec{p}(\lambda)$ unique fixed point of $\Psi$.

3. $\vec{p}(\lambda)$ analytic in $\lambda$. 
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Results conditioned on survival of GW tree

Theorem (P., Spencer)

1. Recall $A[T_0] := \{\exists \text{ a subtree } \cong T_0\}$ for finite $T_0$.

2. Conditioned on survival of $T_\lambda$, each $A[T_0]$ holds almost surely (a.s.). Hence all $A[T_0]$ hold a.s.

3. Equivalence class of every infinite tree (minus a set of measure 0) completely determined by neighbourhood of root of radius $\approx 3^{k+2}$.

4. For all FO properties $A$, of quantifier depth $\leq k$, $P_\lambda[A|\text{survival}] = P_\lambda[A^*|\text{survival}]$, where $A^*$ only depends on this neighbourhood of root.
Probabilities of neighbourhoods of root, conditioned on survival

- Recursive procedure for computing probabilities, recursion on the depth of the neighbourhood.

**Theorem (P., Spencer)**

\[ P_\lambda[\text{neighbourhood} \cong \sigma | \text{survival}] \text{ nice functions of } \lambda \text{ and } p_\lambda, \text{ involving polynomials and iterated exponentials.} \]
Existential monadic second order (EMSO) properties

This part is joint with Dr. Alexander Holroyd, Avi Levy and Prof. Joel Spencer.
EMSO in one line

Allows existential quantification over subsets of vertices. Captures global information.
The scope of EMSO

Definition (Informal EMSO Definition)

∃ an $r$-colouring of the tree such that $P$ holds, where $P$ is FO, involving equality(=), parent-child relationship($\pi$) and colour of the vertices.
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Gives a rule such that the number of children of each colour determine the colour of the parent.
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Example

Colours red and green. A node is green if and only if it has at least one green child. Root is green. Corresponds to survival of the tree.
The Question

1. Survival is EMSO.
2. EMSO involves *only existential* quantifiers over subsets.
3. Finiteness: complement of survival. Only universal quantification so far.
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2. EMSO involves *only existential* quantifiers over subsets.

3. Finiteness: complement of survival. Only universal quantification so far.

4. **Question:** Is finiteness expressible as EMSO?
Finiteness and EMSO

Theorem (P., Spencer)

Finiteness of rooted tree is not expressible as an EMSO property.

Conclusion

The set of EMSO properties is not closed under negation.
Finitness and EMSO

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**A question more interesting to probability:** Could it be that finiteness is an EMSO on all but a subset of trees of measure 0? That is, *can finiteness be almost surely an EMSO?*
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Theorem (Holroyd, Levy, P., Spencer)

Finiteness is not almost surely an EMSO.
Thank you.