A VERY GENERAL QUARTIC DOUBLE FOURFOLD IS NOT STABLY RATIONAL

BRENDAN HASSETT, ALENA PIRUTKA, AND YURI Tschinkel

Abstract. We prove that a very general double cover of the projective four-space, ramified in a quartic threefold, is not stably rational.

1. Introduction

In this note we consider quartic double fourfolds, i.e., hypersurfaces $X_f$ in the weighted projective space $\mathbb{P}(2,1,1,1,1,1)$, with homogeneous coordinates $(s,x,y,z,t,u)$, given by a degree four equation of the form

$$s^2 + f(x,y,z,t,u) = 0.$$  

We work over an uncountable ground field $k$ of characteristic zero. Our main result is

**Theorem 1.** Let $f \in k[x,y,z,t,u]$ be a very general degree four form. Then $X_f$ is not stably rational.

This note is inspired by [Bea15], which used the new technique of the decomposition of the diagonal [Voi15, CTP14, Tot15]. The main difficulty is to construct a special $X$ in the family (1.1) with following properties:

(O) Obstruction: the second unramified cohomology group $H^2_{nr}(X)$ (or another birational invariant) does not vanish,

(R) Resolution: there exists a resolution of singularities $\beta : \tilde{X} \to X$, such that the morphism $\beta$ is universally $\text{CH}_0$-trivial,

(see, e.g., Sections 2 and 4 of [HPT16] for definitions). The verification of both properties for potential examples of $\tilde{X}$ is notoriously difficult. The example considered in [Bea15] satisfies the second property, but not the first.

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Our main goal here is to produce an $X$ satisfying both. We have a candidate example:
\begin{equation}
(1.2) \quad s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2(xy + xz + yz)) = 0.
\end{equation}
The singular locus of $X$ is a connected curve, consisting of 4 components: two nodal cubics, a conic, and a line. How do we find this example? We may transform equation (1.2) to
\begin{equation}
(1.3) \quad yzt^2 + xzt^2 + yu_1^2 + (x^2 + y^2 + z^2 - 2(xy + xz + yz))v_1^2 = 0.
\end{equation}
Precisely, we homogenize via an additional variable $v$, multiply through by $yz$, and absorb the squares into the variables $t_1, u_1,$ and $v_1$. The resulting equation gives a bidegree $(2, 2)$ hypersurface $X'' \subset \mathbb{P}^2 \times \mathbb{P}^3$, birational to $X$ via the coordinate changes. In [HPT16] we proved that this $X''$ satisfies both properties (O) and (R). In particular, $X$ also satisfies (O), since unramified cohomology is a birational invariant.

A direct verification of property (R) for this $X$ is possible, but we found it more transparent to take an alternative approach, applying the specialization argument twice: First we can specialize a very general $X_f$ to a quartic double fourfold $X$ which is singular along a line $\ell$ (contained in the ramification locus); we choose $X$ to be very general subject to this condition. Then we show that the blowup morphism
$$\beta : \tilde{X} := \text{Bl}_\ell(X) \to X$$
is universally $\text{CH}_0$-trivial and that $\tilde{X}$ is smooth, i.e., $X$ satisfies (R). Furthermore, there exists a quadric bundle structure $\pi : \tilde{X} \to \mathbb{P}^2$, with degeneracy divisor a smooth octic curve. In Section 2 we analyze this geometry. We consider a degeneration of these quadric bundles to a fourfold $X'$ which is birational to $X''$, and thus satisfies (O). Its singularities are similar to those considered in [HPT16]; the verification of the required property (R) is easier in this presentation. This is the content of Section 3. In Section 4 we give the argument for failure of stable rationality of very general double covers as in (1.1).

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2. Geometry of quartic double fourfolds

Let $X \to \mathbb{P}^4$ be a double fourfold, ramified along a quartic threefold $Y \subset \mathbb{P}^4$. From the equation (1.1) we see that the quartic double fourfold
$X$ is singular precisely along the singular locus of the quartic threefold $Y \subset \mathbb{P}^4$ given by $f = 0$.

Let $Y$ denote a quartic hypersurface double along $\ell$. These form a linear series of dimension

$$\binom{8}{4} - 5 - 12 = 53$$

and taking into account changes of coordinates—automorphisms of $\mathbb{P}^4$ stabilizing $\ell$—we have 34 free parameters.

Let $\beta : \tilde{X} \rightarrow X$ be the blowup of $X$ along $\ell$. We will analyze its properties by embedding it into natural bundles over $\mathbb{P}^2$.

We start by blowing up $\ell$ in $\mathbb{P}^4$. Projection from $\ell$ gives a projective bundle structure $\varpi : \text{Bl}_{\ell}(\mathbb{P}^4) \rightarrow \mathbb{P}^2$ where we may identify $\text{Bl}_{\ell}(\mathbb{P}^4) \simeq \mathbb{P}(E)$, $E = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$.

Write $h$ for the hyperplane class on $\mathbb{P}^2$ and its pullbacks and $\xi$ for the first Chern class of $\mathcal{O}_{\mathbb{P}(E)}(1)$. Taking global sections $\mathcal{O}_{\mathbb{P}^2} \rightarrow E^\vee$ induces morphisms

$$\mathbb{P}(E) \hookrightarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 5}) \simeq \mathbb{P}^4 \times \mathbb{P}^2;$$

projecting onto the first factor gives the blow up. Its exceptional divisor

$$E \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}) \simeq \mathbb{P}^1 \times \mathbb{P}^2$$

has class $\xi - h$.

Let $\tilde{Y} \subset \mathbb{P}(E)$ denote the proper transform of $Y$, which has class

$$4\xi - 2E = 2\xi + 2h.$$ 

Conversely, divisors in this linear series map to quartic hypersurfaces in $\mathbb{P}^4$ singular along $\ell$. Since $2\xi + 2h$ is very ample in $\mathbb{P}(E)$ the generic such divisor is smooth. The morphism $\varpi$ realizes $\tilde{Y}$ as a conic bundle over $\mathbb{P}^2$; its defining equation $q$ may also be interpreted as a section of the vector bundle $\text{Sym}^2(E^\vee)(2h)$. Let $\gamma : \tilde{Y} \rightarrow Y$ denote the resulting resolution; its exceptional divisor $F = \tilde{Y} \cap E$ is a divisor of bidegree $(0, 2)$ in $E \simeq \mathbb{P}^1 \times \mathbb{P}^2$. Hence $F \rightarrow \ell$ is a trivial conic bundle and $\gamma$ is universally $\text{CH}_0$-trivial.
Let $\tilde{X} \to \mathbb{P}(E)$ denote the double cover branched over $\tilde{Y}$, i.e., $s^2 = q$. This naturally sits in the projectization of an extension

$$0 \to \mathcal{L} \to \mathcal{F} \to E \to 0,$$

where $\mathcal{L}$ is a line bundle. Note the natural maps

$$\text{Sym}^2(E^\vee) \hookrightarrow \text{Sym}^2(F^\vee) \to \mathcal{L}^{-2},$$

and their twists

$$\text{Sym}^2(E^\vee)(2h) \hookrightarrow \text{Sym}^2(F^\vee)(2h) \to \mathcal{L}^{-2}(2h);$$

the last sheaf corresponds to the coordinate $r$. Since we are over $\mathbb{P}^2$ the extension above must split; furthermore, the coordinate $s$ induces a trivialization

$$\mathcal{L}^{-2}(2h) \simeq \mathcal{O}_{\mathbb{P}^2}.$$

Thus we conclude

$$\mathcal{F} \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-1).$$

The divisor $\tilde{X} \subset \mathbb{P}(\mathcal{F})$ is generically smooth; let $\beta : \tilde{X} \to X$ denote the induced resolution of $X$. Its exceptional divisor is a double cover of $E$ branched over $F$, isomorphic to a product of a smooth quadric surface with $\mathbb{P}^1$. (A double cover of $\mathbb{P}^2$ branched along a conic curve is a smooth quadric surface.) It follows that $\beta$ is universally $\text{CH}_0$-trivial.

We summarize the key elements we will need:

**Proposition 2.** Let $X \to \mathbb{P}^4$ be a double fourfold, ramified along a quartic threefold $Y \subset \mathbb{P}^4$. Assume that $Y$ is singular along a line $\ell$ and generic subject to this condition. Let $\beta : \tilde{X} \to X$ be the blowup of $X$ along $\ell$. Then $\tilde{X}$ is smooth and $\beta$ universally $\text{CH}_0$-trivial.

Regarding $\tilde{X} \subset \mathbb{P}(\mathcal{F})$, there is an induced quadric surface fibration

$$\pi : \tilde{X} \to \mathbb{P}^2.$$

Let $D$ denote the degeneracy curve, naturally a divisor in

$$\det(\mathcal{F}^\vee(2h)) \simeq \mathcal{O}_{\mathbb{P}^2}(8).$$

The analysis above gives an explicit determinantal description of the defining equation of $D$. Choose homogeneous forms

$$c \in \Gamma(\mathcal{O}_{\mathbb{P}^2}), \; F_1, F_2, F_3 \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(2)), \; G_1, G_2 \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(3)), \; H \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(4))$$
so that the symmetric matrix associated with $\tilde{X}$ takes the form:

$$
\begin{pmatrix}
  c & 0 & 0 & 0 \\
  0 & F_1 & F_2 & G_1 \\
  0 & F_2 & F_3 & G_2 \\
  0 & G_1 & G_2 & H \\
\end{pmatrix}
$$

We fix coordinates to obtain a concrete equation for $\tilde{X}$. Let $(x, y, z)$ denote coordinates of $\mathbb{P}^2$, or equivalently, linear forms on $\mathbb{P}^4$ vanishing along $\ell$. Let $s$ denote a local coordinate trivializing $\mathcal{O}_{\mathbb{P}^1}(1) \subset \mathcal{F}$, $t$ and $u$ coordinates corresponding to $\mathcal{O}_{\mathbb{P}^1}(2) \subset \mathcal{F}$, and $v$ to $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \mathcal{F}$. Then we have

$$(2.1) \quad \tilde{X} = \{cs^2 + F_1 t^2 + 2F_2 tu + F_3 u^2 + 2G_1 tv + 2G_2 uv + Hv^2 = 0\},$$

where $F_1, F_2, F_3, G_1, G_2,$ and $H$ are homogeneous in $x, y, z$.

Finally, we interpret the degeneration curve in geometric terms. Ignoring the constant, we may write

$$D = (F_1 F_3 - F_2^2) H - F_3 G_1^2 + 2F_2 G_1 G_2 - F_1 G_2^2 = 0.$$  

Modulo $F_1 F_3 - F_2^2$ we have

$$-F_3 G_1^2 + 2F_2 G_1 G_2 - F_1 G_2^2 = 0$$

which is equal to

$$\frac{-1}{F_1} (F_2 G_1 - F_1 G_2)^2 = \frac{-1}{F_3} (F_3 G_1 - F_2 G_2)^2.$$  

Thus we conclude that $D$ is tangent to a quartic plane curve

$$C = \{F_1 F_3 - F_2^2\} = 0$$

at 16 points. Every smooth quartic plane curve admits multiple such representations: Surfaces

$$\{a^2 F_1 + 2ab F_2 + b^2 F_3 = 0\} \subset \mathbb{P}^1_{a,b} \times \mathbb{P}^2$$

are precisely degree two del Pezzo surfaces equipped with a conic bundle structure, the conic structures indexed by non-trivial two-torsion points of the branch curve $C$. One last parameter check: The moduli space of pairs $(C, D)$ consisting of a plane quartic and a plane octic tangent at 16 points depends on

$$14 + 44 - 16 - 8 = 34$$

parameters. This is compatible with our first parameter count.
Remark 3. Smooth divisors $\tilde{X} \subset \mathbb{P}(\mathcal{F})$ as above necessarily have trivial Brauer group. This follows from Pirutka’s analysis [Pir16]: if the degeneracy curve is smooth and irreducible then there cannot be unramified second cohomology. It also follows from a singular version of the Lefschetz hyperplane theorem. Let $\zeta = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$ so that $[\tilde{X}] = 2\zeta + 2h$. This is almost ample: the line bundle $\zeta + h$ contracts the distinguished section $s : \mathbb{P}^2 \to \mathbb{P}(\mathcal{F})$ associated with the summand $\mathcal{O}_{\mathbb{P}(1)} \subset \mathcal{F}$ to a point but otherwise induces an isomorphism onto its image. In particular, $\zeta + h$ induces a small contraction in the sense of intersection homology. The homology version of the Lefschetz Theorem of Goresky-MacPherson [GM88, p. 150] implies that $0 \simeq H^3(\mathbb{P}(\mathcal{F}), \mathbb{Z}) \xrightarrow{\sim} H^3(\tilde{X}, \mathbb{Z})$.

3. Singularities of the special fiber

We specialize (2.1) to:

$$s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2(xy + xz + yz))v^2 = 0.$$  

Proposition 4. The fourfold $X' \subset \mathbb{P}(\mathcal{F})$ defined by (3.1) admits a resolution of singularities $\beta' \colon \tilde{X}' \to X'$ such that $\beta'$ is universally CH$_0$-trivial.

The remainder of this section is a proof of this result.

3.1. The singular locus. A direct computation in Magma (or an analysis as in [HPT16, Section 5]) yields that the singular locus of (3.1) is a connected curve consisting of the following components:

- Singular cubics:
  $$E_z := \{v^2y(y - x)^2 + u^2x = z = s = t = 0\}$$
  $$E_y := \{v^2z(z - x)^2 + t^2x = y = s = u = 0\}$$

- Conics:
  $$R_x := \{u^2 - 4v^2 + t^2 = x = z - y = s = 0\}$$
  $$C_x := \{zu^2 + yt^2 = s = v = x = 0\}$$

The nodes of $E_z$ and $E_y$ are

$$n_z := \{z = s = t = y - x = u = 0\}$$
$$n_y := \{y = s = u = z - x = t = 0\},$$
respectively. Here $R_x$ and $C_x$ intersect transversally at two points,

$$r_{\pm} := \{u \pm it = v = s = z - y = x = 0\}.$$
$R_x$ is disjoint from $E_z$ and $E_y$, and the other curves intersect transversally in a single point (in coordinates $(x, y, z) \times (s, t, u, v)$):

- $E_z \cap E_y = q_x := (1, 0, 0) \times (0, 0, 0, 1)$,
- $E_z \cap C_x = q_y := (0, 1, 0) \times (0, 0, 1, 0)$,
- $E_y \cap C_x = q_z := (0, 0, 1) \times (0, 1, 0, 0)$.

This configuration of curves is similar to the one considered in [HPT16], but the singularities are different.

3.2. Local étale description of the singularities and resolutions. The structural properties of the resolution become clearer after identifying étale normal forms for the singularities.

The main normal form is

\[(3.2) \quad a^2 + b^2 + c^2 = p^2 q^2\]

which is singular along the locus

\[\{a = b = c = p = 0\} \cup \{a = b = c = q = 0\}.\]

This is resolved by successively blowing up along these components in either order. Indeed, after blowing up the first component, using $\{A, B, C, P\}$ for homogeneous coordinates associated with the corresponding generators of the ideal, we obtain

\[A^2 + B^2 + C^2 = P^2 q^2.\]

The exceptional fibers are isomorphic to a non-singular quadric hypersurface (when $q \neq 0$) or a quadric cone (over $q = 0$). Dehomogenizing by setting $P = 1$, we obtain

\[A^2 + B^2 + C^2 = q^2\]

which is resolved by blowing up $\{A = B = C = q = 0\}$. This has ordinary threefold double points at each point, so the exceptional fibers are all isomorphic to non-singular quadric hypersurfaces.

There are cases where

\[\{a = b = c = p = 0\} \cup \{a = b = c = q = 0\}\]

are two branches of the same curve. For example, this could arise from

\[(3.3) \quad a^2 + b^2 + c^2 = (m^2 - n^2 - n^3)^2\]

by setting $p = m - n \sqrt{1 + n}$ and $q = m + n \sqrt{1 + n}$. Of course, we cannot pick one branch to blow up first. We therefore blow up the origin first,
using homogeneous coordinates $A, B, C, D, P, Q$ corresponding to the
generators to obtain

$$A^2 + B^2 + C^2 = P^2 q^2 = Q^2 p^2.$$  

The resulting fourfold is singular along the stratum

$$A = B = C = q = p = 0$$

as well as the proper transforms of the original branches. Indeed, on
dehomogenizing $P = 1$ we obtain local affine equation

$$A^2 + B^2 + C^2 = Q^2 p^2;$$

this is singular along $\{A = B = C = p = 0\}$, the locus where the
exceptional divisor is singular, and $\{A = B = C = q = 0\}$, and proper
transform of $\{a = b = c = q = 0\}$. The local affine equation is the same
as (3.2); we resolve by blowing up the singular locus of the exceptional
divisor followed by blowing up the proper transforms of the branches.
This descends to a resolution of (3.3).

### 3.3. Computation in local charts.

We exploit the symmetry under
the involution exchanging $y \leftrightarrow z$ and $t \leftrightarrow u$. It suffices then to analyze
$E_z, C_x,$ and $R_x$ and the distinguished points $n_z, q_x, q_y,$ and $r_+.$

#### Analysis along the curve $C_x.$

Recall the equation of $X':$

$$s^2 + xy t^2 + xz u^2 + yz(x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)v^2 = 0$$

and the equation of $C_x$: $zu^2 + yt^2 = s = v = x = 0$. We order
coordinates $(x, y, z), (s, t, u, v)$ and write intersections

- $C_x \cap R_x = (0, 1, 1) \times (0, 1, \pm i, 0)$;
- $C_x \cap E_z = (0, 1, 0) \times (0, 0, 1, 0)$;
- $C_x \cap E_y = (0, 0, 1) \times (0, 1, 0, 0)$.

We use the symmetry between $t$ and $u$ to reduce the number of cases.

#### Chart $u = 1, z = 1.$

We extract equations for the exceptional divisor $E$
obtained by blowing up $C_x$. In this chart, $C_x$ takes the form

$$1 + yt^2 = s = v = x = 0$$

and $X'$ is

$$s^2 + x(yt^2 + 1) + v^2(y - 1)^2 + v^2 xG = 0,$$

where $v^2 xG$ are the ‘higher order terms’.

Now we analyse the local charts of the blow up:
the equation for $X'$, up to removing higher order terms, in new coordinates is:

$$s_1^2 + x_1 + v_1^2(y - 1)^2 = 0,$$

this is smooth and rational. The exceptional divisor

$$s_1^2 + x_1 + v_1^2(y - 1)^2 = 0, yt^2 + 1 = 0$$

is rational.

(2) $E: x = 0, s = s_1 x, v = v_1 x, yt^2 + 1 = wx$, equation of $X'$:

$$s_1^2 + w + v_1(y - 1)^2 = 0, yt^2 + 1 = wx,$$

smooth;

(3) $E: s = 0, x = x_1 s, v = v_1 s, yt^2 + 1 = ws$:

$$1 + wx_1 + v_1(y - 1)^2 = 0, yt^2 + 1 = sw,$$

smooth.

(4) $E: v = 0, s = s_1 v, x = x_1 v, yt^2 + 1 = wv$, equation of $X'$ is

$$s_1^2 + wx_1 + (y - 1)^2 = 0, yt^2 + 1 = wv,$$

which has at most ordinary double singularity (corresponding to $C_x \cap R_x = r_\pm$) of type

$$a^2 + b^2 + cd = 0, \ a = b = c = d = 0.$$

This is resolved by one blowup.

Chart $u = 1, y = 1$. In this chart $C_x$ is $z + t^2 = s = v = x = 0$ and $X'$ is

$$s^2 + x(t^2 + z) + v^2(z - 1)^2 + v^2 G = 0,$$

where $v^2 G$ are the 'higher order terms'. We analyze local charts of the blow up:

(1) $E: t^2 + z = 0, s = s_1(t^2 + z), x = x_1(t^2 + z), v = v_1(t^2 + z)$, the equation for $X'$, up to removing higher order terms, in new coordinates is:

$$s_1^2 + x_1 + v_1^2(z - 1)^2 = 0,$$

this is smooth and rational. The exceptional divisor

$$s_1^2 + x_1 + v_1^2(z - 1)^2 = 0, t^2 + z = 0$$

is rational.
(2) $E : x = 0, \ s = s_1 x, \ v = v_1 x, \ t^2 + z = wx$, equation of $X'$:

\[ s_1^2 + w + v_1(z - 1)^2 = 0, \ t^2 + z = wx, \]

smooth;

(3) $E : s = 0, \ x = x_1 s, \ v = v_1 s, \ t^2 + z = ws$:

\[ 1 + wx_1 + v_1(z - 1)^2 = 0, \ t^2 + z = sw, \]

smooth.

(4) $E : v = 0, \ s = s_1 v, \ x = x_1 v, \ t^2 + z = vw$, equation of $X'$ is

\[ s_1^2 + wx_1 + (z - 1)^2 = 0, \ t^2 + z = vw, \]

or, up to removing the higher order terms

\[ s_1^2 + wx_1 + (t^2 + 1)^2 = 0, \ z = -t^2 + vw, \]

this has at most ordinary double singularities

\[ s_1 = w = x_1 = 0, \ t = \pm i \]

(where we meet the proper transform of $R_x$) of type

\[ a^2 + b^2 + cd = 0, \ a = b = c = d \]

resolved as above by one blowup.

**Analysis near $n_z$.** Center the coordinates by setting $\xi = y - 1$

\[ s^2 + (\xi + 1)t^2 + zu^2 + (\xi + 1)z(\xi^2 + z^2 - 2z(\xi + 2)) = 0. \]

Note that $E_z$ is given by

\[ (\xi + 1)\xi^2 + u^2 = z = s = t = 0. \]

We regroup terms

\[ s^2 + (\xi + 1)t^2 + z(u^2 + (\xi + 1)\xi^2) + (\xi + 1)z^2(z - 2\xi - 4) = 0. \]

Provided $\xi \neq -1, -2$ this is étale-locally equal to

\[ s_1^2 + t_1^2 + z_1(u^2 + (\xi + 1)\xi^2) + z_1^2 = 0 \]

which is equivalent to normal form (3.3). When $\xi = -1$ we are at the point $q_x$, which we analyze below. A local computation at $\xi = -2$ shows that the singularity is resolved there by blowing up $E_z$ and the exceptional fiber there is isomorphic to $\mathbb{P}_0$. In other words, we have ordinary threefold double points there as well.
Blowing up the singular point \( n_z \) of \( E_z \). The point \( n_z \) lies in the chart \( x = 1, v = 1 \), where we now make computations. The equation of the point (and the locus we blow up) is

\[
s = t = u = z = y - 1 = 0.
\]

The equation of \( X \) can be written as:

\[
s^2 + yt^2 - 2z^2y(y + 1) + zu^2 + z^3y + yz(y - 1)^2 = 0.
\]

The curve \( E_z \) has equations

\[
y(y - 1)^2 + u^2 = z = s = t = 0.
\]

Now we compute the charts for the blow up:

1. \( E : s = 0 \). The change of variables is \( u = su_1, t = st_1, z = sz_1, y = 1 + y_1s \). Then the equation of \( X' \) (resp. the exceptional divisor \( E \)), up to removing the higher order terms, is:

\[
1 + t_1^2(1 + y_1s) - 2z_1^2(1 + sy_1)(2 + sy_1) = 0
\]

(resp. \( 1 + t_1^2 - 2z_1^2 = 0 \)), so that the blow up and the exceptional divisor are smooth, and \( E \) is rational.

2. \( E : t = 0 \). The change of variables is \( s = st, u = u_1t, z = z_1t, y = 1 + y_1t \): the equations are

\[
s_1^2 + (1 + y_1t) - 2z_1^2(1 + y_1t)(2 + y_1t) = 0,
\]

and \( E \) is given by

\[
s_1^2 + 1 - 4z_1^2 = 0,
\]

so that the blowup is smooth at any point of the exceptional divisor.

3. \( E : z = 0 \), the change of variables is \( s = s_1z, u = u_1z, y = 1 + y_1z \); we obtain

\[
s_1^2 + (1 + y_1z)t_1^2 - 2(1 + y_1z)(2 + y_1z) = 0
\]

and the equation of \( E \) is \( s_1^2 + t_1^2 - 4 = 0 \), so that the blow up is smooth at any point of the exceptional divisor.

4. \( E : y_1 := y - 1 = 0 \), the change of variables is \( z = z_1y_1, s = s_1y_1, u = u_1y_1, t = t_1y_1 \); the equations are

\[
s_1^2 + t_1^2(1+y_1) - 2z_1^2(1+y_1)(2+y_1) + u_1^2y_1z + z_1^3y_1(1+y_1) + z_1(1+y_1) = 0,
\]

this is smooth, as well as the exceptional divisor \( (y_1 = 0) \).
(5) $E : u = 0$, the change of variables is $s = s_1 u, t = t_1 u, z = z_1 u, y = 1 + y_1 u$, the equations for the proper transform of $X'$ are

$$s_1^2 + (1 + y_1 u) t_1^2 - 2 z_1^2 (1 + y_1 u) (1 + y_1 u) + u z_1 + u z_1^2 (1 + y_1 u) + z_1 u y_1^2 (1 + y_1 u) = 0,$$

and the proper transform of $E_z$ is given by

$$(1 + y_1 u) y_1^2 + 1 = z_1 = s_1 = t_1 = 0.$$

The exceptional divisor

$$E : s_1^2 + t_1^2 - 4 z_1^2 = 0$$

is singular along $s_1 = t_1 = z_1 = u = 0$ (and $y_1$ is free). The resulting curve is denoted $R_z \cong \mathbb{P}^1$; note that $R_z$ meets the proper transform of $E_z$ at two points $y_1 = \pm i$.

**Blowing up $R_z$.** For the analysis of singularities we can remove higher order terms, so that the equation of the variety (resp. $R_z$) is given by:

$$s_1^2 + t_1^2 - 4 z_1^2 + u z_1 + u z_1 y_1^2 = 0$$

and $s_1 = t_1 = z_1 = u = 0$.

The charts for the new blow up with exceptional divisor $E'$ are:

(1) $E' : s_1 = 0$, then after the usual change of variables for a blow up, we obtain the equation:

$$1 + t_2^2 - 4 z_2^2 + u z_2 + u z_2 y_1^2 = 0,$$

which is smooth.

(2) $E' : t_1 = 0$ is similar to the previous case.

(3) $E' : z_1 = 0$, we obtain the equation

$$s_2^2 + t_2^2 - 4 + u z_2 + u z_2 y_1^2 = 0,$$

that is smooth;

(4) $E' : u = 0$, we obtain the equation

$$s_2^2 + t_2^2 - 4 z_2^2 + z_2 (1 + y_1^2) = 0,$$

which has ordinary double points at $s_2 = t_2 = z_2 = y_1^2 + 1 = 0$. These are resolved by blowing up the proper transform of $E_z$. 
Analysis near \( q_x \). Dehomogenize
\[
s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2(xy + xz + yz))v^2 = 0
\]
by setting \( v = 1 \) and \( x = 1 \) to obtain
\[
s^2 + yt^2 + zu^2 + yz(1 + y^2 + z^2 - 2(y + z + yz)) = 0.
\]
We first analyze at \( q_x \), the origin in this coordinate system. Note that
\[
1 + y^2 + z^2 - 2(y + z + yz) \neq 0
\]
here and thus its square root can be absorbed (étale locally) in to \( s, t, \) and \( u \) to obtain
\[
s_1^2 + yt_1^2 + zu_1^2 + yz = 0.
\]
Setting \( y_1 = y + u_1^2 \) and \( z_1 = z + t_1^2 \) gives
\[
s_1^2 + y_1z_1 = t_1^2u_1^2,
\]
which is equivalent to normal form (3.2). (The blow up over the generic point of \( E_z \) was analyzed previously.)

Blowing up \( R_x \). Similar to the analysis of singularities near \( R_z \), see also [HPT16, Section 5.2 (4)]

3.4. Summary of the resolution.

Blowup steps. The resolution \( \beta' \) is a sequence of blowups:

1. Blow up the nodes \( n_z \) and \( n_y \); the resulting fourfold is singular along rational curves \( R_z \) and \( R_y \) in the exceptional locus, meeting the proper transforms of \( E_z \) and \( E_y \) transversally in two points sitting over \( n_z \) and \( n_y \), respectively.
2. The exceptional divisors are quadric threefolds singular along \( R_z \) and \( R_y \).
3. At this stage the singular locus consists of six smooth rational curves, the proper transforms of \( E_z, E_y, R_x, C_x \) and the new curves \( R_z \) and \( R_y \), with a total of nine nodes. (This is the configuration appearing in [HPT16, Section 5].)
4. The local analytic structure is precisely as indicated in Section 3.2. Thus we can blow up the six curves in any order to obtain a resolution of singularities. The fibers are either the Hirzebruch surface \( \mathbb{F}_0 \) or a union of Hirzebruch surfaces \( \mathbb{F}_0 \cup \Sigma \mathbb{F}_2 \) where \( \Sigma \simeq \mathbb{P}^1 \) with self intersections \( \Sigma_{\mathbb{F}_0} = 2 \) and \( \Sigma_{\mathbb{F}_2} = -2 \).

For concreteness, we blowup in the order
\[
R_z, R_y, E_z, E_y, C_x, R_x.
\]
Exceptional fibers. The following fibers arise:

- Over the nodes \( n_z \) and \( n_y \): The exceptional fiber has two three-dimensional components. One is the standard resolution of a quadric threefold singular along a line, that is,
  \[
  F' = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_\mathbb{P}(-2)).
  \]
  The other is a quadric surface fibration \( F'' \to \mathbb{P}^1 \), over \( R_z \) and \( R_y \) respectively, smooth except for two fibers corresponding to the intersections with \( E_z \) and \( E_y \); the singular fibers are unions \( F_0 \cup F_2 \) as indicated above. The intersection \( F' \cap F'' \) is along the distinguished subbundle
  \[
  \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}) \subset F'
  \]
  which meets the smooth fibers of \( F'' \to \mathbb{P}^1 \) in hyperplanes and the singular fibers in smooth rational curves in \( F_2 \) with self-intersection 2.

- Over \( E_z \): the exceptional divisor is a quadric surface fibration over \( \mathbb{P}^1 \), with two degenerate fibers of the form \( F_0 \cup F_2 \) corresponding to the intersections with \( C_x \) and \( E_y \).

- Over \( E_y \): the exceptional divisor is a quadric surface fibration with one degenerate fiber, corresponding to the intersection with \( C_x \).

- Over \( C_x \): a quadric surface fibration with two degenerate fibers corresponding to the intersections with \( R_x \).

- Over \( R_x \): a smooth quadric surface fibration.

In each case, the fibers of \( \beta' \) are universally \( \text{CH}_0 \)-trivial.

4. Proof of Theorem 1

We recall implications of the “integral decomposition of the diagonal and specialization” method, following [CTP14], [Voi15].

Theorem 5. [Voi15, Theorem 2.1], [CTP14, Theorem 1.14 and Theorem 2.3] Let

\[
\phi : \mathcal{X} \to B
\]

be a flat projective morphism of complex varieties with smooth generic fiber. Assume that there exists a point \( b \in B \) so that the fiber

\[
X := \phi^{-1}(b)
\]

satisfies the following conditions:
• $X$ admits a desingularization

$$\beta : \tilde{X} \to X,$$

where the morphism $\beta$ is universally $\text{CH}_0$-trivial,

• $\tilde{X}$ is not universally $\text{CH}_0$-trivial.

Then a very general fiber of $\phi$ is not universally $\text{CH}_0$-trivial and, in particular, not stably rational.

We apply this twice:

1. As mentioned in the introduction, $X''$ satisfies property (O); this is an application of Pirutka’s computation of the unramified second cohomology of quadric surface bundles over $\mathbb{P}^2$ \cite{Pir16}. As we saw in the introduction, $X'$ is birational to $X''$. Section 3.4 yields property (R) for $X'$. We conclude that the very general hypersurfaces $\tilde{X} \subset \mathbb{P}(F)$ described in Section 2, following Proposition 2, fail to be universally $\text{CH}_0$-trivial.

2. By Proposition 2, the map $\beta : \tilde{X} \to X$ is universally $\text{CH}_0$-trivial. A second application of Theorem 5 to the family of double fourfolds ramified along a quartic threefold completes the proof of Theorem 1.

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Department of Mathematics, Brown University, Box 1917 151 Thayer Street Providence, RI 02912, USA
E-mail address: bhassett@math.brown.edu

Courant Institute, New York University, New York, NY 10012, USA
E-mail address: pirutka@cims.nyu.edu

Courant Institute, New York University, New York, NY 10012, USA
E-mail address: tschinkel@cims.nyu.edu

Simons Foundation, 160 Fifth Avenue, New York, NY 10010, USA