A VERY GENERAL QUARTIC DOUBLE FOURFOLD IS NOT STABLY RATIONAL

BRENDAN HASSETT, ALENA PIRUTKA, AND YURI TSCHINKEL

Abstract. We prove that a very general double cover of the projective four-space, ramified in a quartic threefold, is not stably rational.

1. Introduction

In this note we consider quartic double fourfolds, i.e., hypersurfaces X_f in the weighted projective space $\mathbb{P}(2,1,1,1,1,1)$, with homogeneous coordinates (s,x,y,z,t,u), given by a degree four equation of the form

$$s^2 + f(x,y,z,t,u) = 0. \tag{1.1}$$

We work over an uncountable ground field k of characteristic zero. Our main result is

Theorem 1. *Let $f \in k[x,y,z,t,u]$ be a very general degree four form. Then X_f is not stably rational.*

This note is inspired by [Bea15], which used the new technique of the decomposition of the diagonal [Voi15, CTP14, Tot15]. The main difficulty is to construct a special X in the family (1.1) with following properties:

- (**O**) Obstruction: the second unramified cohomology group $H^2_{nr}(X)$ (or another birational invariant) does not vanish,
- (**R**) Resolution: there exists a resolution of singularities $\beta : \tilde{X} \to X$, such that the morphism β is universally CH_0-trivial,

(see, e.g., Sections 2 and 4 of [HPT16] for definitions). The verification of both properties for potential examples of X is notoriously difficult. The example considered in [Bea15] satisfies the second property, but not the first.

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Our main goal here is to produce an X satisfying both. We have a candidate example:

$$s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2(xy + xz + yz)) = 0. \tag{1.2}$$

The singular locus of X is a connected curve, consisting of 4 components: two nodal cubics, a conic, and a line. How do we find this example? We may transform equation (1.2) to

$$yzs^2 + xzt_1^2 + xyu_1^2 + (x^2 + y^2 + z^2 - 2(xy + xz + yz))v_1^2 = 0. \tag{1.3}$$

Precisely, we homogenize via an additional variable v, multiply through by yz, and absorb the squares into the variables t_1, u_1, and v_1. The resulting equation gives a bidegree $(2, 2)$ hypersurface

$$X'' \subset \mathbb{P}^2 \times \mathbb{P}^3,$$

birational to X via the coordinate changes. In [HPT16] we proved that this X'' satisfies both properties (O) and (R). In particular, X also satisfies (O), since unramified cohomology is a birational invariant.

A direct verification of property (R) for this X is possible, but we found it more transparent to take an alternative approach, applying the specialization argument twice: First we can specialize a very general X_f to a quartic double fourfold X which is singular along a line ℓ (contained in the ramification locus); we choose X to be very general subject to this condition. Then we show that the blowup morphism

$$\beta : \tilde{X} := \mathrm{Bl}_\ell(X) \to X$$

is universally CH_0-trivial and that $\tilde{X}$ is smooth, i.e., X satisfies (R). Furthermore, there exists a quadric bundle structure $\pi : \tilde{X} \to \mathbb{P}^2$, with degeneracy divisor a smooth octic curve. In Section 2 we analyze this geometry. We consider a degeneration of these quadric bundles to a fourfold X' which is birational to X'', and thus satisfies (O). Its singularities are similar to those considered in [HPT16]; the verification of the required property (R) is easier in this presentation. This is the content of Section 3. In Section 4 we give the argument for failure of stable rationality of very general double covers as in (1.1).

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2. Geometry of quartic double fourfolds

Let $X \to \mathbb{P}^4$ be a double fourfold, ramified along a quartic threefold $Y \subset \mathbb{P}^4$. From the equation (1.1) we see that the quartic double fourfold

X is singular precisely along the singular locus of the quartic threefold $Y \subset \mathbb{P}^4$ given by $f = 0$.

Let Y denote a quartic hypersurface double along ℓ. These form a linear series of dimension

$$\binom{8}{4} - 5 - 12 = 53$$

and taking into account changes of coordinates—automorphisms of $\mathbb{P}^4$ stabilizing ℓ—we have 34 free parameters.

Let $\beta : \tilde{X} \to X$ be the blowup of X along ℓ. We will analyze its properties by embedding it into natural bundles over $\mathbb{P}^2$.

We start by blowing up ℓ in $\mathbb{P}^4$. Projection from ℓ gives a projective bundle structure

$$\varpi : \mathrm{Bl}_\ell(\mathbb{P}^4) \to \mathbb{P}^2$$

where we may identify

$$\mathrm{Bl}_\ell(\mathbb{P}^4) \simeq \mathbb{P}(\mathcal{E}), \quad \mathcal{E} = \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1).$$

Write h for the hyperplane class on $\mathbb{P}^2$ and its pullbacks and ξ for the first Chern class of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Taking global sections

$$\mathcal{O}_{\mathbb{P}^2}^{\oplus 5} \twoheadrightarrow \mathcal{E}^\vee$$

induces morphisms

$$\mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus}) \simeq \mathbb{P}^4 \times \mathbb{P}^2;$$

projecting onto the first factor gives the blow up. Its exceptional divisor

$$E \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}) \simeq \mathbb{P}^1 \times \mathbb{P}^2$$

has class $\xi - h$.

Let $\tilde{Y} \subset \mathbb{P}(\mathcal{E})$ denote the proper transform of Y, which has class

$$4\xi - 2E = 2\xi + 2h.$$

Conversely, divisors in this linear series map to quartic hypersurfaces in $\mathbb{P}^4$ singular along ℓ. Since $2\xi + 2h$ is very ample in $\mathbb{P}(\mathcal{E})$ the generic such divisor is smooth. The morphism ϖ realizes $\tilde{Y}$ as a conic bundle over $\mathbb{P}^2$; its defining equation q may also be interpretted as a section of the vector bundle $\mathrm{Sym}^2(\mathcal{E}^\vee)(2h)$. Let $\gamma : \tilde{Y} \to Y$ denote the resulting resolution; its exceptional divisor $F = \tilde{Y} \cap E$ is a divisor of bidegree $(0, 2)$ in $E \simeq \mathbb{P}^1 \times \mathbb{P}^2$. Hence $F \to \ell$ is a trivial conic bundle and γ is universally CH_0-trivial.

Let $\tilde{X} \to \mathbb{P}(\mathcal{E})$ denote the double cover branched over $\tilde{Y}$, i.e., $s^2 = q$. This naturally sits in the projectization of an extension

$$0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{E} \to 0,$$

where $\mathcal{L}$ is a line bundle. Note the natural maps

$$\mathrm{Sym}^2(\mathcal{E}^\vee) \hookrightarrow \mathrm{Sym}^2(\mathcal{F}^\vee) \twoheadrightarrow \mathcal{L}^{-2},$$

and their twists

$$\mathrm{Sym}^2(\mathcal{E}^\vee)(2h) \hookrightarrow \mathrm{Sym}^2(\mathcal{F}^\vee)(2h) \twoheadrightarrow \mathcal{L}^{-2}(2h);$$

the last sheaf corresponds to the coordinate r. Since we are over $\mathbb{P}^2$ the extension above must split; furthermore, the coordinate s induces a trivialization

$$\mathcal{L}^{-2}(2h) \simeq \mathcal{O}_{\mathbb{P}^2}.$$

Thus we conclude

$$\mathcal{F} \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1).$$

The divisor $\tilde{X} \subset \mathbb{P}(\mathcal{F})$ is generically smooth; let $\beta : \tilde{X} \to X$ denote the induced resolution of X. Its exceptional divisor is a double cover of E branched over F, isomorphic to a product of a smooth quadric surface with $\mathbb{P}^1$. (A double cover of $\mathbb{P}^2$ branched along a conic curve is a smooth quadric surface.) It follows that β is universally CH_0-trivial.

We summarize the key elements we will need:

Proposition 2. *Let $X \to \mathbb{P}^4$ be a double fourfold, ramified along a quartic threefold $Y \subset \mathbb{P}^4$. Assume that Y is singular along a line ℓ and generic subject to this condition. Let $\beta : \tilde{X} \to X$ be the blowup of X along ℓ. Then $\tilde{X}$ is smooth and β universally* CH_0*-trivial.*

Regarding $\tilde{X} \subset \mathbb{P}(\mathcal{F})$, there is an induced quadric surface fibration

$$\pi : \tilde{X} \to \mathbb{P}^2.$$

Let D denote the degeneracy curve, naturally a divisor in

$$\det(\mathcal{F}^\vee(2h)) \simeq \mathcal{O}_{\mathbb{P}^2}(8).$$

The analysis above gives an explicit determinantal description of the defining equation of D. Choose homogeneous forms

$$c \in \Gamma(\mathcal{O}_{\mathbb{P}^2}),\ F_1, F_2, F_3 \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(2)),\ G_1, G_2 \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(3)), H \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(4))$$

so that the symmetric matrix associated with $\tilde{X}$ takes the form:

$$\begin{pmatrix} c & 0 & 0 & 0 \\ 0 & F_1 & F_2 & G_1 \\ 0 & F_2 & F_3 & G_2 \\ 0 & G_1 & G_2 & H \end{pmatrix}$$

We fix coordinates to obtain a concrete equation for $\tilde{X}$. Let (x, y, z) denote coordinates of $\mathbb{P}^2$, or equivalently, linear forms on $\mathbb{P}^4$ vanishing along ℓ. Let s denote a local coordinate trivializing $\mathcal{O}_{\mathbb{P}^1}(1) \subset \mathcal{F}$, t and u coordinates corresponding to $\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \subset \mathcal{F}$, and v to $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \mathcal{F}$. Then we have

$$\tilde{X} = \{cs^2 + F_1t^2 + 2F_2tu + F_3u^2 + 2G_1tv + 2G_2uv + Hv^2 = 0\}, \tag{2.1}$$

where F_1, F_2, F_3, G_1, G_2, and H are homogeneous in x, y, z.

Finally, we interpret the degeneration curve in geometric terms. Ignoring the constant, we may write

$$D = (F_1F_3 - F_2^2)H - F_3G_1^2 + 2F_2G_1G_2 - F_1G_2^2 = 0.$$

Modulo $F_1F_3 - F_2^2$ we have

$$-F_3G_1^2 + 2F_2G_1G_2 - F_1G_2^2 = 0$$

which is equal to

$$\frac{-1}{F_1}(F_2G_1 - F_1G_2)^2 = \frac{-1}{F_3}(F_3G_1 - F_2G_2)^2.$$

Thus we conclude that D is tangent to a quartic plane curve

$$C = \{F_1F_3 - F_2^2\} = 0$$

at 16 points. *Every* smooth quartic plane curve admits multiple such representations: Surfaces

$$\{a^2F_1 + 2abF_2 + b^2F_3 = 0\} \subset \mathbb{P}^1_{a,b} \times \mathbb{P}^2$$

are precisely degree two del Pezzo surfaces equipped with a conic bundle structure, the conic structures indexed by non-trivial two-torsion points of the branch curve C. One last parameter check: The moduli space of pairs (C, D) consisting of a plane quartic and a plane octic tangent at 16 points depends on

$$14 + 44 - 16 - 8 = 34$$

parameters. This is compatible with our first parameter count.

Remark 3. *Smooth* divisors $\tilde{X} \subset \mathbb{P}(\mathcal{F})$ as above necessarily have trivial Brauer group. This follows from Pirutka's analysis [Pir16]: if the degeneracy curve is smooth and irreducible then there cannot be unramified second cohomology. It also follows from a singular version of the Lefschetz hyperplane theorem. Let $\zeta = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$ so that $[\tilde{X}] = 2\zeta + 2h$. This is almost ample: the line bundle $\zeta + h$ contracts the distinguished section $s : \mathbb{P}^2 \to \mathbb{P}(\mathcal{F})$ associated with the summand $\mathcal{O}_{\mathbb{P}^1}(1) \subset \mathcal{F}$ to a point but otherwise induces an isomorphism onto its image. In particular, $\zeta + h$ induces a small contraction in the sense of intersection homology. The homology version of the Lefschetz Theorem of Goresky-MacPherson [GM88, p. 150] implies that $0 \simeq H^3(\mathbb{P}(\mathcal{F}), \mathbb{Z}) \xrightarrow{\sim} H^3(\tilde{X}, \mathbb{Z})$.

3. Singularities of the special fiber

We specialize (2.1) to:

$$s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2(xy + xz + yz))v^2 = 0. \tag{3.1}$$

Proposition 4. *The fourfold $X' \subset \mathbb{P}(\mathcal{F})$ defined by (3.1) admits a resolution of singularities $\beta' : \tilde{X}' \to X'$ such that β' is universally* CH_0*-trivial.*

The remainder of this section is a proof of this result.

3.1. **The singular locus.** A direct computation in `Magma` (or an analysis as in [HPT16, Section 5]) yields that the singular locus of (3.1) is a connected curve consisting of the following components:

- Singular cubics:

$$E_z := \{v^2y(y-x)^2 + u^2x = z = s = t = 0\}$$
$$E_y := \{v^2z(z-x)^2 + t^2x = y = s = u = 0\}$$

- Conics:

$$R_x := \{u^2 - 4v^2 + t^2 = x = z - y = s = 0\}$$
$$C_x := \{zu^2 + yt^2 = s = v = x = 0\}$$

The nodes of E_z and E_y are

$$\mathfrak{n}_z := \{z = s = t = y - x = u = 0\}$$
$$\mathfrak{n}_y := \{y = s = u = z - x = t = 0\},$$

respectively. Here R_x and C_x intersect transversally at two points,

$$\mathfrak{r}_\pm := \{u \pm it = v = s = z - y = x = 0\};$$

R_x is disjoint from E_z and E_y, and the other curves intersect transversally in a single point (in coordinates $(x, y, z) \times (s, t, u, v)$):

$$\begin{aligned} E_z \cap E_y &= \mathfrak{q}_x := (1,0,0) \times (0,0,0,1), \\ E_z \cap C_x &= \mathfrak{q}_y := (0,1,0) \times (0,0,1,0), \\ E_y \cap C_x &= \mathfrak{q}_z := (0,0,1) \times (0,1,0,0). \end{aligned}$$

This configuration of curves is similar to the one considered in [HPT16], but the singularities are different.

3.2. **Local étale description of the singularities and resolutions.** The structural properties of the resolution become clearer after identifying étale normal forms for the singularities.

The main normal form is

$$a^2 + b^2 + c^2 = p^2 q^2 \tag{3.2}$$

which is singular along the locus

$$\{a = b = c = p = 0\} \cup \{a = b = c = q = 0\}.$$

This is resolved by successively blowing up along these components in either order. Indeed, after blowing up the first component, using $\{A, B, C, P\}$ for homogeneous coordinates associated with the corresponding generators of the ideal, we obtain

$$A^2 + B^2 + C^2 = P^2 q^2.$$

The exceptional fibers are isomorphic to a non-singular quadric hypersurface (when $q \neq 0$) or a quadric cone (over $q = 0$). Dehomogenizing by setting $P = 1$, we obtain

$$A^2 + B^2 + C^2 = q^2$$

which is resolved by blowing up $\{A = B = C = q = 0\}$. This has ordinary threefold double points at each point, so the exceptional fibers are all isomorphic to non-singular quadric hypersurfaces.

There are cases where

$$\{a = b = c = p = 0\} \cup \{a = b = c = q = 0\}$$

are two branches of the same curve. For example, this could arise from

$$a^2 + b^2 + c^2 = (m^2 - n^2 - n^3)^2 \tag{3.3}$$

by setting $p = m - n\sqrt{1+n}$ and $q = m + n\sqrt{1+n}$. Of course, we cannot pick one branch to blow up first. We therefore blow up the origin first, using homogeneous coordinates A, B, C, D, P, Q corresponding to the generators to obtain

$$A^2 + B^2 + C^2 = P^2q^2 = Q^2p^2.$$

The resulting fourfold is singular along the stratum

$$A = B = C = q = p = 0$$

as well as the proper transforms of the original branches. Indeed, on dehomogenizing $P = 1$ we obtain local affine equation

$$A^2 + B^2 + C^2 = Q^2p^2;$$

this is singular along $\{A = B = C = p = 0\}$, the locus where the exceptional divisor is singular, and $\{A = B = C = Q = 0\}$, and proper transform of $\{a = b = c = q = 0\}$. The local affine equation is the same as (3.2); we resolve by blowing up the singular locus of the exceptional divisor followed by blowing up the proper transforms of the branches. This descends to a resolution of (3.3).

3.3. **Computation in local charts.** We exploit the symmetry under the involution exchanging $y \leftrightarrow z$ and $t \leftrightarrow u$. It suffices then to analyze E_z, C_x, and R_x and the distinguished points $\mathfrak{n}_z$, $\mathfrak{q}_x$, $\mathfrak{q}_y$, and $\mathfrak{r}_+$.

Analysis along the curve C_x. Recall the equation of X':

$$s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)v^2 = 0$$

and the equation of C_x: $zu^2 + yt^2 = s = v = x = 0$. We order coordinates $(x, y, z), (s, t, u, v)$ and write intersections

- $C_x \cap R_x = (0, 1, 1) \times (0, 1, \pm i, 0)$;
- $C_x \cap E_z = (0, 1, 0) \times (0, 0, 1, 0)$;
- $C_x \cap E_y = (0, 0, 1) \times (0, 1, 0, 0)$.

We use the symmetry between t and u to reduce the number of cases.

Chart $u = 1$, $z = 1$. We extract equations for the exceptional divisor $\mathbf{E}$ obtained by blowing up C_x. In this chart, C_x takes the form

$$1 + yt^2 = s = v = x = 0$$

and X' is

$$s^2 + x(yt^2 + 1) + v^2(y - 1)^2 + v^2xG = 0,$$

where v^2xG are the 'higher order terms'.

Now we analyse the local charts of the blow up:

(1) $\mathbf{E}: yt^2+1=0$, $s=s_1(yt^2+1), x=x_1(yt^2+1), v=v_1(yt^2+1)$, the equation for X', up to removing higher order terms, in new coordinates is:
$$s_1^2+x_1+v_1^2(y-1)^2=0,$$
this is smooth and rational. The exceptional divisor
$$s_1^2+x_1+v_1^2(y-1)^2=0, yt^2+1=0$$
is rational.

(2) $\mathbf{E}: x=0$, $s=s_1x, v=v_1x, yt^2+1=wx$, equation of X':
$$s_1^2+w+v_1(y-1)^2=0, yt^2+1=wx,$$
smooth;

(3) $\mathbf{E}: s=0$, $x=x_1s, v=v_1s, yt^2+1=ws$:
$$1+wx_1+v_1(y-1)^2=0, yt^2+1=sw,$$
smooth.

(4) $\mathbf{E}: v=0, s=s_1v, x=x_1v, yt^2+1=wv$, equation of X' is
$$s_1^2+wx_1+(y-1)^2=0, yt^2+1=wv,$$
which has at most ordinary double singularity (corresponding to $C_x\cap R_x=\mathfrak{r}_\pm$) of type
$$a^2+b^2+cd=0,\ a=b=c=d=0.$$
This is resolved by one blowup.

Chart $u=1$, $y=1$. In this chart C_x is $z+t^2=s=v=x=0$ and X' is
$$s^2+x(t^2+z)+v^2(z-1)^2+v^2xG=0,$$
where v^2xG are the 'higher order terms'. We analyze local charts of the blow up:

(1) $\mathbf{E}: t^2+z=0$, $s=s_1(t^2+z), x=x_1(t^2+z), v=v_1(t^2+z)$, the equation for X', up to removing higher order terms, in new coordinates is:
$$s_1^2+x_1+v_1^2(z-1)^2=0,$$
this is smooth and rational. The exceptional divisor
$$s_1^2+x_1+v_1^2(z-1)^2=0, t^2+z=0$$
is rational.

(2) $\mathbf{E}: x=0,\ s=s_1x, v=v_1x, t^2+z=wx$, equation of X':
$$s_1^2+w+v_1(z-1)^2=0, t^2+z=wx,$$
smooth;

(3) $\mathbf{E}: s=0,\ x=x_1s, v=v_1s, t^2+z=ws$:
$$1+wx_1+v_1(z-1)^2=0, t^2+z=sw,$$
smooth.

(4) $\mathbf{E}: v=0, s=s_1v, x=x_1v, t^2+z=vw$, equation of X' is
$$s_1^2+wx_1+(z-1)^2=0, t^2+z=wv,$$
or, up to removing the higher order terms
$$s_1^2+wx_1+(t^2+1)^2=0, z=-t^2+wv,$$
this has at most ordinary double singularities
$$s_1=w=x_1=0, t=\pm i$$
(where we meet the proper transform of R_x) of type
$$a^2+b^2+cd=0, a=b=c=d$$
resolved as above by one blowup.

Analysis near $\mathfrak{n}_z$. Center the coordinates by setting $\xi=y-1$
$$s^2+(\xi+1)t^2+zu^2+(\xi+1)z(\xi^2+z^2-2z(\xi+2))=0.$$
Note that E_z is given by
$$(\xi+1)\xi^2+u^2=z=s=t=0.$$
We regroup terms
$$s^2+(\xi+1)t^2+z(u^2+(\xi+1)\xi^2)+(\xi+1)z^2(z-2\xi-4)=0.$$
Provided $\xi\neq -1,-2$ this is étale-locally equal to
$$s_1^2+t_1^2+z_1(u^2+(\xi+1)\xi^2)+z_1^2=0$$
which is equivalent to normal form (3.3). When $\xi=-1$ we are at the point $\mathfrak{q}_x$, which we analyze below. A local computation at $\xi=-2$ shows that the singularity is resolved there by blowing up E_z and the exceptional fiber there is isomorphic to $\mathbb{F}_0$. In other words, we have ordinary threefold double points there as well.

Blowing up the singular point $\mathfrak{n}_z$ of E_z. The point $\mathfrak{n}_z$ lies in the chart $x = 1, v = 1$, where we now make computations. The equation of the point (and the locus we blow up) is

$$s = t = u = z = y - 1 = 0.$$

The equation of X can be written as:

$$s^2 + yt^2 - 2z^2y(y+1) + zu^2 + z^3y + yz(y-1)^2 = 0.$$

The curve E_z has equations

$$y(y-1)^2 + u^2 = z = s = t = 0.$$

Now we compute the charts for the blow up:

(1) $\mathbf{E} : s = 0$. The change of variables is $u = su_1, t = st_1, z = sz_1, y = 1 + y_1s$. Then the equation of X' (resp. the exceptional divisor $\mathbf{E}$), up to removing the higher order terms, is:

$$1 + t_1^2(1 + y_1s) - 2z_1^2(1 + sy_1)(2 + sy_1) = 0$$

(resp. $1 + t_1^2 - 2z_1^2 = 0$), so that the blow up and the exceptional divisor are smooth, and $\mathbf{E}$ is rational.

(2) $\mathbf{E} : t = 0$. The change of variables is $s = s_1t, u = u_1t, z = z_1t, y = 1 + y_1t$: the equations are

$$s_1^2 + (1 + y_1t) - 2z_1^2(1 + y_1t)(2 + y_1t) = 0,$$

and E is given by

$$s_1^2 + 1 - 4z_1^2 = 0,$$

so that the blowup is smooth at any point of the exceptional divisor.

(3) $\mathbf{E} : z = 0$, the change of variables is $s = s_1z, u = u_1z, y = 1 + y_1z$; we obtain

$$s_1^2 + (1 + y_1z)t_1^2 - 2(1 + y_1z)(2 + y_1z) = 0$$

and the equation of $\mathbf{E}$ is $s_1^2 + t_1^2 - 4 = 0$, so that the blow up is smooth at any point of the exceptional divisor.

(4) $\mathbf{E} : y_1 := y - 1 = 0$, the change of variables is $z = z_1y_1, s = s_1y_1, u = u_1y_1, t = t_1y_1$; the equations are

$$s_1^2 + t_1^2(1+y_1) - 2z_1^2(1+y_1)(2+y_1) + u_1^2y_1z + z_1^3y_1(1+y_1) + z_1(1+y_1) = 0,$$

this is smooth, as well as the exceptional divisor ($y_1 = 0$).

(5) $\mathbf{E}: u = 0$, the change of variables is $s = s_1u, t = t_1u, z = z_1u, y = 1 + y_1u$, the equations for the proper transform of X' are

$$s_1^2+(1+y_1u)t_1^2-2z_1^2(1+y_1u)(2+y_1u)+uz_1+uz_1^3(1+y_1u)+z_1uy_1^2(1+y_1u) = 0,$$

and the proper transform of E_z is given by

$$(1 + y_1u)y_1^2 + 1 = z_1 = s_1 = t_1 = 0.$$

The exceptional divisor

$$\mathbf{E} : s_1^2 + t_1^2 - 4z_1^2 = 0$$

is singular along $s_1 = t_1 = z_1 = u = 0$ (and y_1 is free). The resulting curve is denoted $R_z \simeq \mathbb{P}^1$; note that R_z meets the proper transform of E_z at two points $y_1 = \pm i$.

Blowing up R_z. For the analysis of singularities we can remove higher order terms, so that the equation of the variety (resp. R_z) is given by:

$$s_1^2 + t_1^2 - 4z_1^2 + uz_1 + uz_1y_1^2 = 0$$

and $s_1 = t_1 = z_1 = u = 0$.

The charts for the new blow up with exceptional divisor $\mathbf{E}'$ are:

(1) $\mathbf{E}' : s_1 = 0$, then after the usual change of variables for a blow up, we obtain the equation:

$$1 + t_2^2 - 4z_2^2 + u_2z_2 + u_2z_2y_1^2 = 0,$$

which is smooth.

(2) $\mathbf{E}' : t_1 = 0$ is similar to the previous case.

(3) $\mathbf{E}' : z_1 = 0$, we obtain the equation

$$s_2^2 + t_2^2 - 4 + u_2 + u_2y_1^2 = 0,$$

that is smooth;

(4) $\mathbf{E}' : u = 0$, we obtain the equation

$$s_2^2 + t_2^2 - 4z_2^2 + z_2(1 + y_1^2) = 0,$$

which has ordinary double points at $s_2 = t_2 = z_2 = y_1^2 + 1 = 0$. These are resolved by blowing up the proper transform of E_z.

Analysis near $\mathfrak{q}_x$. Dehomogenize

$$s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2(xy + xz + yz))v^2 = 0$$

by setting $v = 1$ and $x = 1$ to obtain

$$s^2 + yt^2 + zu^2 + yz(1 + y^2 + z^2 - 2(y + z + yz)) = 0.$$

We first analyze at $\mathfrak{q}_x$, the origin in this coordinate system. Note that $1 + y^2 + z^2 - 2(y + z + yz) \neq 0$ here and thus its square root can be absorbed (étale locally) in to s, t, and u to obtain

$$s_1^2 + yt_1^2 + zu_1^2 + yz = 0.$$

Setting $y_1 = y + u_1^2$ and $z_1 = z + t_1^2$ gives

$$s_1^2 + y_1 z_1 = t_1^2 u_1^2,$$

which is equivalent to normal form (3.2). (The blow up over the generic point of E_z was analyzed previously.)

Blowing up R_x. Similar to the analysis of singularities near R_z, see also [HPT16, Section 5.2 (4)]

3.4. **Summary of the resolution.**

Blowup steps. The resolution β' is a sequence of blowups:

(1) Blow up the nodes $\mathfrak{n}_z$ and $\mathfrak{n}_y$; the resulting fourfold is singular along rational curves R_z and R_y in the exceptional locus, meeting the proper transforms of E_z and E_y transversally in two points sitting over $\mathfrak{n}_z$ and $\mathfrak{n}_y$, respectively.
(2) The exceptional divisors are quadric threefolds singular along R_z and R_y.
(3) At this stage the singular locus consists of six smooth rational curves, the proper transforms of E_z, E_y, R_x, C_x and the new curves R_z and R_y, with a total of nine nodes. (This is the configuration appearing in [HPT16, Section 5].)
(4) The local analytic structure is precisely as indicated in Section 3.2. Thus we can blow up the six curves in any order to obtain a resolution of singularities. The fibers are either the Hirzebruch surface $\mathbb{F}_0$ or a union of Hirzebruch surfaces $\mathbb{F}_0 \cup_\Sigma \mathbb{F}_2$ where $\Sigma \simeq \mathbb{P}^1$ with self intersections $\Sigma^2_{\mathbb{F}_0} = 2$ and $\Sigma^2_{\mathbb{F}_2} = -2$.

For concreteness, we blowup in the order

$$R_z, R_y, E_z, E_y, C_x, R_x.$$

Exceptional fibers. The following fibers arise:

- Over the nodes $\mathfrak{n}_z$ and $\mathfrak{n}_y$: The exceptional fiber has two three-dimensional components. One is the standard resolution of a quadric threefold singular along a line, that is,
$$\mathbf{F}' = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)).$$
The other is a quadric surface fibration $\mathbf{F}'' \to \mathbb{P}^1$, over R_z and R_y respectively, smooth except for two fibers corresponding to the intersections with E_z and E_y; the singular fibers are unions $\mathbb{F}_0 \cup \mathbb{F}_2$ as indicated above. The intersection $\mathbf{F}' \cap \mathbf{F}''$ is along the distinguished subbundle
$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}) \subset \mathbf{F}'$$
which meets the smooth fibers of $\mathbf{F}'' \to \mathbb{P}^1$ in hyperplanes and the singular fibers in smooth rational curves in $\mathbb{F}_2$ with self-intersection 2.
- Over E_z: the exceptional divisor is a quadric surface fibration over $\mathbb{P}^1$, with two degenerate fibers of the form $\mathbb{F}_0 \cup \mathbb{F}_2$ corresponding to the intersections with C_x and E_y.
- Over E_y: the exceptional divisor is a quadric surface fibration with one degenerate fiber, corresponding to the intersection with C_x.
- Over C_x: a quadric surface fibration with two degenerate fibers corresponding to the intersections with R_x.
- Over R_x: a smooth quadric surface fibration.

In each case, the fibers of β' are universally CH_0-trivial.

4. Proof of Theorem 1

We recall implications of the "integral decomposition of the diagonal and specialization" method, following [CTP14], [Voi15].

Theorem 5. [Voi15, Theorem 2.1], [CTP14, Theorem 1.14 and [Theorem 2.3] *Let*
$$\phi : \mathcal{X} \to B$$
be a flat projective morphism of complex varieties with smooth generic fiber. Assume that there exists a point $b \in B$ so that the fiber
$$X := \phi^{-1}(b)$$
satisfies the following conditions:

- *X admits a desingularization*

$$\beta : \tilde{X} \to X,$$

where the morphism β is universally CH_0*-trivial,*
- *$\tilde{X}$ is not universally* CH_0*-trivial.*

Then a very general fiber of ϕ is not universally CH_0*-trivial and, in particular, not stably rational.*

We apply this twice:

(1) As mentioned in the introduction, X'' satisfies property (O); this is an application of Pirutka's computation of the unramified second cohomology of quadric surface bundles over $\mathbb{P}^2$ [Pir16]. As we saw in the introduction, X' is birational to X''. Section 3.4 yields property (R) for X'. We conclude that the very general hypersurfaces $\tilde{X} \subset \mathbb{P}(\mathcal{F})$ described in Section 2, following Proposition 2, fail to be universally CH_0-trivial.
(2) By Proposition 2, the map $\beta : \tilde{X} \to X$ is universally CH_0-trivial. A second application of Theorem 5 to the family of double fourfolds ramified along a quartic threefold completes the proof of Theorem 1.

References

[Bea15] Arnaud Beauville. A very general quartic double fourfold or fivefold is not stably rational. *Algebr. Geom.*, 2(4):508–513, 2015.

[CTP14] Jean-Louis Colliot-Thélène and Alena Pirutka. Hypersurfaces quartiques de dimension 3 : non rationalité stable. *Ann. Sci. ENS*, to appear, 2014. `arXiv:1402.4153`.

[GM88] Mark Goresky and Robert MacPherson. *Stratified Morse theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988.

[HPT16] Brendan Hassett, Alena Pirutka, and Yuri Tschinkel. Stable rationality quadric surface bundles over surfaces, 2016. `arXiv:1603.09262`.

[Pir16] Alena Pirutka. Varieties that are not stably rational, zero-cycles and unramified cohomology, 2016. `arXiv:1603.09261`.

[Tot15] Burt Totaro. Hypersurfaces that are not stably rational. *Journal of the American Mathematical Society*, to appear, 2015. `arXiv:1502.04040`.

[Voi15] Claire Voisin. Unirational threefolds with no universal codimension 2 cycle. *Invent. Math.*, 201(1):207–237, 2015.

Department of Mathematics, Brown University, Box 1917 151 Thayer Street Providence, RI 02912, USA
E-mail address: `bhassett@math.brown.edu`

Courant Institute, New York University, New York, NY 10012, USA
E-mail address: `pirutka@cims.nyu.edu`

Courant Institute, New York University, New York, NY 10012, USA
E-mail address: `tschinkel@cims.nyu.edu`

Simons Foundation, 160 Fifth Avenue, New York, NY 10010, USA