Diagonal arithmetics : exercises

1. Examples of vanishing of the Chow group of zero-cycles:

(a) Let $X$ be a smooth projective retract rational variety over a field $k$. Show that $X$ is $\text{CH}_0$-trivial. (Hint: one can use a moving lemma here.)

(b) Let $X$ be a smooth projective rationally connected variety over an algebraically closed field $k$. Show that $\text{CH}_0(X) = 0$.

2. Proof of the moving lemma for zero-cycles. Let $k$ be an infinite perfect field and let $X$ be a smooth irreducible quasi-projective variety over $k$. Let $U \subset X$ be a nonempty Zariski open of $X$. Let $F = X \setminus U$.

(a) Show that if $\dim X = 1$, then any zero-cycle in $X$ is rationally equivalent to a zero-cycle with support in $U$. (One can use that the semi-local ring $\mathcal{O}_{X,F}$ is a principal ideal domain).

(b) Let $d = \dim X$. Let $x \in X$ be a closed point. Show that there is a function $g \in \mathcal{O}_{X,x}$ such that the condition $g = 0$ defines locally a closed subset containing $F$.

(c) Show that one can find a system $f_1, \ldots, f_{d-1}$ of regular parameters of $\mathcal{O}_{X,x}$ such that the image of $g$ in $\mathcal{O}_{X,x}/(f_1, \ldots, f_{d-1})$ is nonzero.

(d) Let $C$ be a curve defined as a closure in $X$ of the locus $f_1 = \ldots = f_{d-1} = 0$. Let $\pi : D \to X$ be the normalisation of $C$. Show that there exists a point $y \in D$ such that $x = \pi_*(y)$.

(e) use the case of $\dim X = 1$ to show that $x$ is rationally equivalent to a zero-cycle supported on $U$. Conclude that any zero-cycle on $X$ is rationally equivalent to a zero-cycle supported on $U$.

3. $\text{CH}_0$-universal triviality and conditions on fibers. Let $k$ be a field and let $f : Z \to Y$ be a proper map between algebraic varieties over $k$. Assume that for any $M$ a (scheme) point of $Y$, the fiber $Z_M$ is $\text{CH}_0$-universally trivial.

(a) Show that the push-forward $f_* : \text{CH}_0(Z) \to \text{CH}_0(Y)$ is surjective.

(b) Let $z \in \ker(f_*)$. Show that there exists some integral curves $C_i \subset Y$ closed in $Y$ such that $f_*(z) = \sum \text{div}_{\hat{C}_i}(g_i)$ for some functions $g_i$ on $C_i$ ($\hat{C}_i$ is the normalization of $C_i$).

(c) Show that one can find finite surjective maps $f^j_i : D^j_i \to C_i$ from integral curves $D^j_i \subset Z$ such that $\sum_j n^j_i \text{deg}(f^j_i) = 1$ for some $n^j_i \in \mathbb{Z}$.

(d) Consider $z' = z - \sum_i \sum_j n^j_i \text{div}_{D^j_i}(g_i)$. Show that $f_*(z') = 0$ as a zero-cycle. Deduce that $z'$ is a sum of finitely many zero-cycles of degree zero included in fibers of $f$. Conclude that $z$ is rationally equivalent to zero.

4. Stable birational invariance of the unramified cohomology groups:
(a) Let $k$ be a field and let $F \subset A^1_k$ be a finite subset of an affine line over $k$. Show that there exists an exact sequence

$$0 \to H^i(k,\mu_n^{\otimes j}) \to H^i(A^1_k \setminus F,\mu_n^{\otimes j}) \to \bigoplus_{P \in F} H^{i-1}(k(P),\mu_n^{\otimes (j-1)}) \to 0$$

(hint: one can use purity and Gysin exact sequence)

(b) Deduce that the following sequence is exact:

$$0 \to H^i(k,\mu_n^{\otimes j}) \to H^i(k(t),\mu_n^{\otimes j}) \to \bigoplus_{P \in A^1_k} H^{i-1}(k(P),\mu_n^{\otimes (j-1)}) \to 0$$