THE POISSON EQUATION

Warning: We will only perform in this section formal manipulations ("a priori estimates" i.e. estimates assuming enough regularity). A rigorous justification, introducing the concept of weak solution, can be done as in the heat equation section.

We consider the Poisson equation $\Delta u = 0$ with data on a hyperplane. More precisely, we solve

$$\begin{align*}
\Delta u &= 0 \quad \text{on} \quad \mathbb{R}^{d+1} = \{(x_1, \ldots, x_d, t), \ t > 0\}, \\
n(x, 0) &= f \quad \text{on} \quad \{t = 0\}.
\end{align*}$$

Note: we adopted coordinates $(x_1, \ldots, x_d, t)$ on $\mathbb{R}^{d+1}$, for a reason which will soon become clear.

The equation can be written

$$\begin{align*}
\{ (\partial^2_t + \Delta_x) u = 0 \quad \text{for} \quad t > 0 \\
n(t = 0) &= f
\end{align*}$$

Taking the Fourier transform in $x$,

$$\begin{align*}
\{ (\partial^2_t + 15^2) \hat{u}(t, \xi) &= 0 \\
\hat{u}(t = 0) &= \hat{f}
\end{align*}$$
Therefore, \( \hat{u}(t, \xi) = a \) linear combination of \( e^{t|\xi|} \) and \( e^{-t|\xi|} \), for each \( \xi \).

To discard \( e^{t|\xi|} \), we add the boundary condition at infinity \( u(t) \to 0 \) as \( t \to \infty \).

Therefore, \( \hat{u}(t, \xi) = e^{-t|\xi|} \hat{f}(\xi) \)

Since \( \mathcal{F}(e^{-t|\xi|}) = \frac{Ca}{(1 + |\xi|^2 \frac{dt}{2})} \) (the constant \( Ca \) makes sure that \( \int \frac{Ca}{(1 + |\xi|^2 \frac{dt}{2})} d\omega = 1 \))

\[ u(t) = \frac{1}{2t} \frac{Ca}{(1 + \frac{1}{t^2 t^2} \frac{dt}{2})} \ast f \]

We deduce the estimates

\[ \text{Theorem (i) (Positivity)} \quad u(t) \geq 0 \text{ if } f \geq 0 \]

(ii) (Integrability) \( \|u(t)\|_{L^1} \leq \frac{C}{t^{a(t/4 - 1)}} \|f\|_{L^1} \)

(iii) (Regularity) \( \|\nabla^k u\|_{H^s} \leq \frac{C}{t^{k}} \|f\|_{H^s} \)
THE WAVE EQUATION

Warning. We simply prove a priori estimates.

The equation

\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \Delta u &= 0 \\
u(t=0) &= u_0 \\
\partial_n u(t=0) &= n_0 \\
u &= u(t, x)
\end{aligned}
\]

Fourier representation $x \rightarrow \xi$

\[
\frac{\partial^2 \hat{u}}{\partial \xi^2} + |\xi|^2 \hat{u} = 0
\]

\[
\hat{u}(t=0) = \hat{u}_0
\]

\[
\partial_n \hat{u}(t=0) = \hat{n}_0
\]

Solving this ODE,

\[
\hat{u}(t, \xi) = \cos(t|\xi|) \hat{u}_0(\xi) + \sin(t|\xi|) \hat{n}_0(\xi)
\]

Energy estimate. Multiplying by $\partial_t u$ and integrating:

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \left( |\nabla u|^2 + |\partial_t u|^2 \right) dx = 0.
\]
What is a "good" nonlinear problem?

In most physical setups, everything starts with an energy $E(u)$. Its gradient $\nabla E(u)$ is such that $\forall \varphi$,

$$E(u + \varepsilon \varphi) = E(u) + \varepsilon \nabla E(u) \cdot \varphi + O(\varepsilon^2)$$

(everything is formal for the moment; to make things concrete, you can think of $E$ as a functional over $\mathbb{R}^d$).

The following problems arise:

1. **Critical points of $E$**: They satisfy
   $$\nabla E(u) = 0.$$

2. **Gradient flow of $E$**: A steepest descent
   $$\dot{t} u = -\nabla E(u).$$

   Obviously, $E$ decreases along solutions of this flow:
   $$\frac{d}{dt} E(u) = \nabla E(u) \cdot \dot{t} u = - (\nabla E(u))^2$$

   Observe that critical points of $E$ are stationary solutions of its gradient flow.
3 Hamiltonian flow of $E$.

Given an anti-self-adjoint operator $J$ s.t. $J^2 = -I_d$,

$$\partial_t u = J \nabla E(u)$$

This flow conserves the energy:

$$\frac{d}{dt} E(u) = \nabla E(u) \cdot J \nabla E(u) = 0.$$ 

That was the (very) general setup! We will consider one of the simplest examples:

$$E(u) = \int_{\mathbb{R}^d} 1 \nabla u \cdot \nabla \phi \, dx + \int_{\mathbb{R}^d} |u|^4 \, dx \quad u \in H^1 \cap L^4.$$ 

Then \( E(u + \varepsilon \phi) = E(u) + 2 \varepsilon \int \nabla u \cdot \nabla \phi \, dx \)

$$+ 4 \varepsilon \int |u|^3 \phi \, dx + O(\varepsilon^2)$$

$$= E(u) + \varepsilon \int (-2 \Delta u + 4u^3) \phi \, dx + O(\varepsilon^2)$$

We deduce that \( \nabla E(u) = -2 \Delta u + 4u^3 \)

① The critical points of $E$ solve the nonlinear Poisson equation \(-\Delta u + u^3 = 0 \quad \text{(NLP)}\)

② The gradient flow of $E$ becomes
The nonlinear heat equation

\[ (NLH) \quad \partial_t u - \Delta u + u^3 = 0. \]

(3) The simplest Hamiltonian flow is obtained by thinking of \( u \) as a complex variable, and choosing \( J = i \).
(you also need to modify the scalar product, but let us forget about that).
One obtains the nonlinear Schrödinger equation

\[ (NLS) \quad i \partial_t u - \Delta u + u^3 = 0. \]

(4) A close cousin is the nonlinear wave equation

\[ \partial_t^2 u - \Delta u + u^3 = 0. \]
WELL-POSEDNESS (in the sense of Hadamard)

Consider an evolution problem, and Banach spaces $X$ (for the data) and $Y$ (for the solution).
We say that initial value problem (evolution equation with prescribed data at $t=0$) is well-posed if

i) Existence: for any $u_0 \in X$, there exists a solution $y \in Y$.

ii) Uniqueness: this solution is unique.

iii) Continuous dependence: the map $J_{u_0} : X \rightarrow Y$ is continuous.

Local well-posedness if $Y$ is local in time (e.g. $C([0,1], X)$)

Global well-posedness if $Y$ is global in time
Theorem (LWP for the nonlinear heat equation)

Consider \( (NLH) \)
\[
\begin{cases}
\partial_t u - \Delta u = u^3 \\
\mu(t, x) \in \mathbb{R} \times [0, T] \times \mathbb{T}^d \\
\mu(t=0) = u_0 \in \mathbb{R} \times [0, T] \times \mathbb{R}^d
\end{cases}
\]

It is LWP for data in \( H^N(\mathbb{T}^d) \) with \( N \) sufficiently big, namely:

for \( u_0 \in H^N \), there exists \( T > 0 \) and a unique solution \( u \in C([0, T], H^N) \). Furthermore, the map
\[
(0, T) \times H^N \rightarrow C([0, T], H^N)
\]
is continuous.

Remarks
1. The topology on \( C([0, T], H^N) \) is given by \( \|u\| = \sup_{0 < t < T} \|u(t)\|_{H^N} \).
2. LWP is, in a sense, the least one can ask in order to study the dynamical properties of \( (NLH) \) or any other PDE. But LWP and its proof do not say much about the actual behavior of solutions.
3. Indeed, the proof that will be given works identically:
   * on \( \mathbb{R}^d \) instead of \( \mathbb{T}^d \).
• Changing $u^3$ into $-u^3$, or $F(u)$, for $F$ smooth, with $F'(0) = 0$

• Replacing $f u - \Delta u$ by $i \bar{f} u - \Delta u$, or $\partial_t u - \Delta u$ (minimal change)

$\Rightarrow$ LWP does not capture fine properties of the equation.

4. The proof will proceed by a fixed-point argument.

But it is useful to think of this fixed-point argument as a rigorous formalization of an a-priori estimate.

PDE Wisdom: look for the a priori estimate! It can "always" be turned into a rigorous existence proof.
The \textit{a-priori} estimate

We make a qualitative assumption on the solution \((u)\) is smooth\) and deduce a quantitative assumption.

The smoothness of \(u\) justifies the following manipulations:

- Take \(N\) derivatives of the eq.
- Multiply the eq. by \(\nabla^N u\)
- Integrate over \(\Omega\)
- Integrate by parts

This gives, denoting \(\nabla^N\) for \(\partial^{2N}\), with \(|\alpha| = N\)

\[
\int \partial_t \nabla^N u \cdot \nabla^N u \, dx = \int \Delta \nabla^N u \cdot \nabla^N u = \int \nabla^N u \cdot \nabla^N u^3
\]

\[
\frac{1}{2} \frac{d}{dt} \int |\nabla^N u|^2 \, dx + \int |\nabla \nabla^N u|^2 \, dx \quad (\ast)
\]

\((\ast)\) = \(\int \nabla^N u \cdot \nabla^N u^3 \, dx \leq \|u\|_{H^N} \|u^3\|_{H^N} \leq C \|u\|_{H^N}^4 \) (by the algebra property of \(H^N\))

Therefore:

\[
\frac{d}{dt} \|\nabla^N u\|_{L^2}^2 \leq C \|u\|_{H^N}^4
\]
Similarly, \( \frac{d}{dt} \| u \|_2^2 \leq C \| u \|_{H^2}^4 \)

We find \( \frac{d}{dt} \| u \|_{H^2}^2 \leq C \| u \|_{H^2}^5 \)

If \( X = \| u \|_{H^2}^2 \), \( \frac{dX}{dt} \leq CX^2 \)

Easy to solve: \( \frac{dX}{X^2} \leq C \, dt \)

\(- \frac{1}{X(t)} + \frac{1}{X(0)} \leq Ct \)

\( \Rightarrow X(t) \leq \frac{1}{X(0) - Ct} \)