HW 5

1. Prove that, if $f \in C(\mathbb{R}^d)$ (i.e., $f$ continuous on $\mathbb{R}^d$),
$$
(e^{t\Delta} f)(x) \to f(x) \text{ for all } x \in \mathbb{R}^d,
$$
and that the convergence is uniform on compact intervals.

2. Prove that, if $f \in L^p(\mathbb{R}^d)$, $e^{t\Delta} f \overset{t \to 0}{\to} f$ in $L^p$,
if $1 \leq p < \infty$.

3. Show that this fails if $p = \infty$.

2. We want to study the linear Schrödinger equation
$$
\begin{cases}
\partial_t u - \Delta u = 0 & \text{set on } \mathbb{R} \times \mathbb{R}^d \\
\quad u(t=0) = f
\end{cases}
$$

2. Consider first the regularized model
$$
\begin{cases}
(e^{-i\epsilon t}) \partial_t u - \Delta u = 0 \\
\quad u(t=0) = f
\end{cases}
$$

and find an expression for the solution as convolution with a kernel $K_\epsilon(t,x)$. We admit that solutions of the linear Schrödinger equation are given by $u(t) = K(t) * f$, and denote $u(t) = K(t) * f = e^{it \Delta} f$.

4. Write down the weak formulation.

4. Show that, for data in $L^q$,
$$
\|e^{it\Delta} f\|_{L^2} = \|f\|_{L^2} \\
\|e^{it\Delta} f\|_{L^\infty} \leq \frac{C}{t^{d/2}} \|f\|_{L^1}
$$
2. Show that, for data in $L^2$, $e^{i\Delta} f$ does not preserve positivity and that a smoothing estimate of the type $\|e^{i\Delta} f\|_{H^s} \leq C(t) \|f\|_{L^2}$ cannot hold.

3. We consider the heat equation with compactly supported, smooth data, on $\mathbb{R}^d$ or $\mathbb{T}^d$.
   a. Show that, on $\mathbb{T}^d$, $e^{t\Delta} f \to \int_{\mathbb{T}^d} f$ as $t \to \infty$ uniformly. How fast does the convergence take place? (argue in Fourier space)
   b. (Harder) On $\mathbb{R}^d$, show that
      \[ e^{t\Delta} f(x) \sim \left( \int_{\mathbb{R}^d} f \right) \frac{1}{t^{d/2}} G \left( \frac{x}{\sqrt{t}} \right) \]
      in an appropriate topology.