HW 3-4  (Due Feb 21)

1. We want to prove the Hardy inequality.
   \[ \left( \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} \, dx \right)^{1/2} \leq \frac{2}{d-2} \left( \int_{\mathbb{R}^d} |\nabla f|^2 \, dx \right)^{1/2} \quad \text{if } d \geq 3 \]

   a) Does it follow from the Sobolev embedding theorem?
   b) Check that the scaling is satisfied (consider \( f_{\lambda} \))
   c) Can it hold for \( d=1,2 \) if \( f \in C_0^\infty \)?
   d) Prove this identity for \( f \in C_0^\infty \) by using the identity \( (x \cdot \nabla) |x|^{-2} = -2 |x|^{-2} (x \cdot \nabla = \sum x_i \partial_i) \)
   e) Extend this result to any \( f \in H^1(\mathbb{R}^d), d \geq 3 \)

2. Assume that \( \varphi \in C_0^\infty \), with \( |\text{Jac } \varphi| = \left| \det \frac{\partial \varphi}{\partial x} \right| < 1 \), and let \( \psi : \mathbb{R}^d \to \mathbb{R}^d \)
   \[ x \mapsto x + \varphi(x) \]
   Prove then that \( f \circ \psi \in H^1(\mathbb{R}^d) \) if \( f \in H^1(\mathbb{R}^d) \).
   What are optimal conditions on \( \varphi \) for such a result to hold?
3. Prove that the $H^s(\mathbb{R}^d)$ norm is equivalent to the following norm:
\[
\left( \int_{\mathbb{R}^d} |u(x)|^2 \, dx \right)^{1/2} + \left( \iint_{|y| > 2s} \frac{|u(x+y) - u(x)|^2}{|y|^{d+2s}} \, dx \, dy \right)^{1/2}
\]
if $s \in (0,1)$ and $d \geq 2$. [Argue in Fourier space]

4. We want to prove the Rellich-Kondrakov theorem.
   Let $K$ be a compact set of $\mathbb{R}^d$.
   a) For $s > 0$, $\varepsilon > 0$, $s - \varepsilon > 0$, show that the embedding $H^s(\mathbb{R}^d) \subset H^{s-\varepsilon}(\mathbb{R}^d)$ is not compact (i.e., there are bounded sequences in $H^s(\mathbb{R}^d)$ which do not admit any convergent subsequence in $H^{s-\varepsilon}$).
   b) If $0 < s < \frac{d}{2}$, $\frac{2s}{d-2s}$, show that the embedding $H^s(K) \subset L^2(\mathbb{R}^d)$ is not compact.
   c) Prove that $H^s(K) \subset H^{s-\varepsilon}(\mathbb{R}^d)$ (compact embedding) [Hint: show that it suffices to find a subsequence $f_n$ s.t. $f_n \rightharpoonup f$ uniformly on $B(0,N)$ for all $N$, and then use Arzela-Ascoli].
   d) Prove that $H^0(T^d) \subset H^{0-\varepsilon}(T^d)$ [Argue in Fourier space; the definition of $H^0(T^d)$ is identical to that of $H^0(\mathbb{R}^d)$ replacing the continuous frequencies by discrete frequencies $k$, and $\int_{\mathbb{R}^d}$ by $\sum_{\mathbb{Z}^d}$].