The Shape of the Ideal Column

Steven J. Cox

The column stands both as the essence of an architectural order and as the first flexible body to fall to mathematical analysis. The aesthetic ideal, formulated and realized by the ancient Greeks, was recorded by Vitruvius in *De Architectura* (*circa* 25 B.C.). The result, a subtle variation on the cylindrical profile, calls for a bulge at approximately one third of the column's height and a diminution near its top. With a denunciation of this aesthetic ideal, Lagrange in 1773 formulated the first scientific criterion, one based on strength rather than appearance. A number of missteps in applying the calculus led him to the mistaken conclusion that the cylinder was the strongest hinged column. Though T. Clausen, in 1851, appeared to succeed where Lagrange had failed, C. Truesdell, troubled by "elements of mystery" remaining a century later, invited a fresh approach. In response, J. Keller recovered, in greater generality, the result of Clausen. Keller published his findings in 1960 and with I. Tadjbakhsh in a paper of 1962 tackled the remaining boundary conditions of interest. M. Overton and I have recently closed the longstanding debate over Tadjbakhsh and Keller's claim that the strongest clamped and clamped-hinged columns possess interior points where the cross section vanishes.

Here I trace the influence exerted by the aesthetic ideal through the early stages of the theory of elasticity and the subsequent formulation of the scientific ideal under the influence of Euler and Laugier. I indicate where the extension of Keller's successful analysis breaks down, tracing the cause to the lack of differentiability, indeed the lack of continuity, of Lagrange's measure of strength. Finally, in a setting in which an optimal design exists, I discuss the role of double eigenvalues and the consequent need for nonsmooth analysis in the construction of necessary conditions.

The Aesthetic Ideal

The swelling of columns was but one of the optical refinements employed by the Greeks to counter perceived imperfections. This practice, which varied to a degree dictated by the proposed structure's size and surroundings, reached its zenith in the Parthenon where

> The delicate curves and inclinations of the horizontal and vertical lines include the rising curves given to the stylobate and entablature in order to impart a feeling of life and to prevent the appearance of sagging, the convex curve to which the entasis of the columns was worked in order to correct the optical illusion of concavity which might have resulted if the sides had been straight, and the slight inward inclinations of the axes of the columns so as to give

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> the whole building an appearance of greater strength; all entailed a mathematical precision in the setting out of the work and in its execution which is probably unparalleled in the world [5, p. 178].

For Vitruvius these refinements were direct consequences of the principle: *Ergo quod oculus fallit, ratiocinatione est exequendum.* "For what the eye cheats us of must be made up by calculation" [19, v. 1 p. 179]. Its application to the design of columns induced Vitruvius to warn that ". . . the sight follows gracious contours; and unless we flatter its pleasure, by proportionate alterations of the modules (so that by adjustment there is added the amount to which it suffers illusion), an uncouth and ungracious aspect will be presented to the spectators. As to the swelling which is made in the middle of the columns (this among the Greeks is called entasis), an illustrated formula will be furnished at the end of the book to show how the entasis may be done in a graceful and appropriate manner" [19, v. 1 p. 179]. That close inspection has turned up "no Roman columns without an entasis" [15, p. 121] suggests such warnings were indeed heeded.

Though Vitruvius's illustration was lost, his text on this point differs so little from Alberti's discussion of entasis in *De re Aedictoria* (1450) that one expects the illustration (Figure 1) in Bartoli's 1550 Italian translation of this work to faithfully represent the ideal of Vitruvius. This despite Alberti's claim that his prescription "is not a discovery of the ancients handed down in some writing, but what we have noted ourselves, by careful and studious observation of the work of the best architects. What follows principally concerns the rules of lineaments; it is of the greatest importance, and may give great delight to painters" [1, p. 188]. It must be noted that here Alberti abandons the rationale of optical refinement for his much more abstract notion of lineaments ("the correct, infallible way of joining and fitting together those lines and angles which define and enclose the surfaces of the building" [1, p. 7]) and so obscures the motivation behind entasis. In addition, as with Vitruvius, Alberti's wooden prescription fails to encompass the full range of Greek examples, from the lack of entasis in the Temple of Apollo at Corinth to its overabundance in the Basilica at Paestrum. Though the correction of optical illusions is surely at work in these structures, the existence of a single theory embracing all cases "is liable to serious objections" [13, p. 103].

Subsequent architects, though keenly aware of the optical refinements as practiced, appear ignorant of, or at least unconcerned with, the causes that induced them. In the writings of the 16th-century Italian architects Palladio and Vignola, for example, one finds detailed illustrated prescriptions of entasis without discussion of the condition for which this remedy is being prescribed. Divorced from its inspiration the practice of entasis suffered instances of both exaggeration,

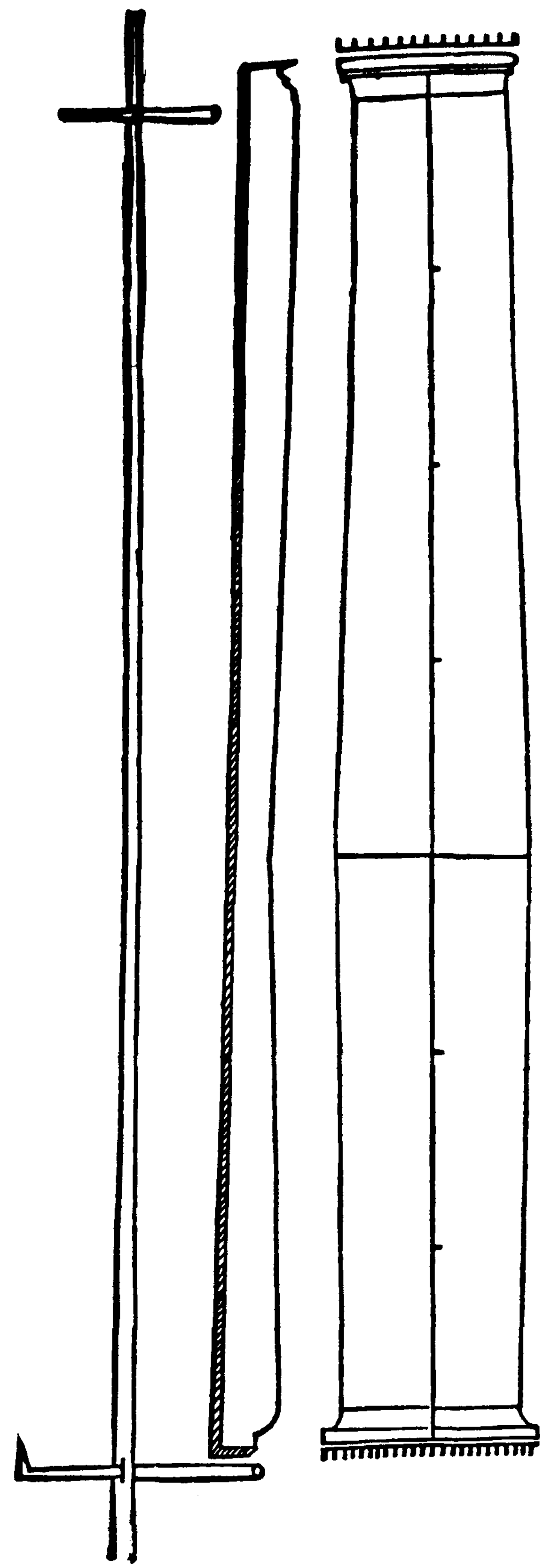

Figure 1. Illustration of entasis in Bartoli's 1550 Italian translation of *De re Aedictoria*.

"even to a cigar shape" [5, p. 186], and neglect, "too delicate an ornament to be appreciated by the common man; columns more often were tailored to follow the form of the perfect cylinder" [18, v. 3, p. 495]. If this suggests a waning of the influence of Vitruvius, the decline of the Baroque would signal a return to the Greek models and the man in whose writings they were preserved. M. Blondel, the director of Louis XIV's Royal Academy of Architecture and member of the Paris Royal Academy of Sciences, endowed his 1675 treatise on architecture with the subtitle *L'origine & les Principes d'Architecture, & les practiques des cinq Ordres suivant la doctrine de Vitruve.* Blondel's treatment of entasis, differing from that of Palladio or Vignola in his attempt to express it analytically, coincided with the announcements of R. Hooke, of the Royal Society of London, regarding both "The true Mathematical and Mechanical form of all manner of Arches for Building" and "The true Theory of Elasticity."

Early Work on Elasticity

In his treatise on elasticity of 1678 [7] Hooke writes, "The Power of any Spring is in the same proportion with the Tension thereof. . . . The same will be found, if trial be made, with a piece of dry wood that will bend and return, if one end thereof be fixt in a horizontal posture, and to the other end be hanged weights to make it bend downwards." The latter remark, in stating the column's restoring force in terms of the load-induced strain, contains the seed of the first constitutive law for a flexible body. It would remain for Euler and the Bernoullis to quantify the relevant notions of stress and strain and so flesh out this bending law of Hooke's.

Clearly aware of the three-dimensional nature of the column, James Bernoulli, beginning in 1691, nonetheless sought to describe its bending in terms of the planar deformation of a "neutral axis," γ. In particular, associating the strain in the column with the curvature κ of γ and the column's stress with the bending moment M, he attempted to derive Hooke's law, $M \propto \kappa$. This program proved too ambitious, indeed the position of the neutral axis eludes us to this day, and it was not until 1732 that James's nephew Daniel Bernoulli first postulated $M \propto \kappa$ in a theory of bending. Euler, in an unpublished work on a special case, identified this proportion with the product of E, the modulus of extension, and I, the second moment of area of the column's cross section about a line through its centroid normal to the plane of bending, with the result

$$M = EI\kappa. \tag{1}$$

This is known as the Bernoulli-Euler formula for the bending of a column, while the name of Thomas Young, but three years old when Euler produced its precise definition, is that typically attached to the modulus E. In accordance with these measures of stress and strain D. Bernoulli, in a letter of 1738, posed to Euler the problem of finding that curve for which the stored energy $\int_0^\ell M\,\kappa\,ds$ was a minimum.

In *Additamentum I de curvis elasticis* (1744) [6, s. 1 v. 24], an appendix to his text on the calculus of variations, Euler subsequently solved the problem of the inextensible elastica, i.e., for constant E and I he found that curve of prescribed length with prescribed terminal displacements and slopes and minimum stored energy. Here it will suffice to consider curves that are graphs of functions over the interval $[0, \ell]$. In this context, Euler succeeded in minimizing

$$\int_0^\ell \frac{EI|u''|^2}{(1+|u'|^2)^{9/4}}\,dx \;-\; \lambda \int_0^\ell (1+|u'|^2)^{1/2}\,dx, \tag{2}$$

where u and u' are prescribed at 0 and at ℓ and λ is the Lagrange multiplier associated with the length constraint. Identifying λ with the axial load necessary to sustain a prescribed deformation, Euler found the precise load under which an initially straight column would commence to bend. This value, now known as the Euler buckling (or critical) load, is $\lambda_c = EI\pi^2/(4f^2)$, where f is the length of a quarter period of the deformed curve. Euler singled out the hinged case, where the displacement and moment vanish at each end, for which $f = \ell/2$ and so

$$\lambda_c = EI\,\frac{\pi^2}{\ell^2}\,.$$

That the quadratures required to obtain this result owed their existence to the constant nature of E and I perhaps led Euler to the alternative characterization of λ_c in *Sur la force des colonnes* (1757) [6, s. 2 v. 17]. In this work he observed, again in the context of hinged ends, that as λ_c marks the load under which deformation begins one could indeed restrict attention to the linearization of the first variation of (2) about the straight state. It follows that λ_c is the least eigenvalue of

$$\begin{gathered}(EIu'')'' + \lambda u'' = 0,\\ u(0) = EIu''(0) = u(\ell) = EIu''(\ell) = 0.\end{gathered} \tag{3}$$

The corresponding first eigenfunction, u_c, as above measures displacement of the neutral axis and contributes to the bending moment M_c via $M_c = EIu_c''$. This choice of boundary conditions proved especially convenient, for in this case u_c and M_c are each first eigenfunctions (with first eigenvalue λ_c) of

$$EIy'' + \lambda y = 0, \quad y(0) = y(\ell) = 0, \tag{4}$$

and hence $u_c = M_c$. With this formulation Euler pro-

Figure 2. The primitive hut: frontispiece from the second edition of the *Essai sur l'Architecture*, engraved by Ch. Eisen.

ceeded to compute λ_c for the class of nonuniform columns in which $E \equiv 1$ and $I(x) = (a + bx/\ell)^q$ at selected values of q. For columns with circular cross section, I is proportional to the square of the cross sectional area, A. The case $q = 2$, for which Euler finds

$$\lambda_c = \frac{b^2}{\ell^2}\left(\frac{1}{4} + \frac{\pi^2}{(\log(1 + b/a))^2}\right), \tag{5}$$

then corresponds to columns of circular cross section for which A either increases or decreases linearly with length. He indeed calculated λ_c for a number of other exponents but stopped short of formulating a basis of comparison with which to distinguish the various choices.

The Scientific Ideal

This task was taken up by Lagrange in *Sur la figure des colonnes* (1773) [9, v. 2]. Lagrange sought to maximize λ_c, suitably normalized, over solids of revolution with prescribed length. In particular, he sought that function for which the "relative strength"

$$\frac{\lambda_c(A)}{V^2(A)} \tag{6}$$

achieves its maximum. Here $A : [0, \ell] \rightarrow [0, \infty)$ measures cross-sectional area, $V(A) = \int_0^\ell A\,dx$ is the column's volume, and $\lambda_c(A)$ is the least eigenvalue of (4) with $E = 1$ and $I = A^2$. With this I in (4) it follows for every positive α that $\lambda_c(\alpha A) = \alpha^2\lambda_c(A)$, and, as the volume obeys $V(\alpha A) = \alpha V(A)$, if $\hat{A}$ maximizes (6) then so too does $\alpha\hat{A}$. Consequently, maximizing (6) is in fact equivalent to maximizing λ_c over solids of revolution of prescribed volume and length.

Rather than arguing the efficacy of his relative strength in the design of columns, Lagrange instead attacks the legitimacy of the aesthetic ideal of the Greeks. Seeking to upstage Vitruvius, "le législateur des architectes modernes," Lagrange claimed in his search for a rationale underlying the prescription of entasis to find nothing more sound than a resemblance to the human body, a profile he found, with reference to the primitive hut, inferior to that of the trunk of a tree. Noting the loss of Vitruvius's original illustration, Lagrange then denounced the prescriptions of Palladio, Vignola, and Blondel as arbitrary variations on an already shaky theme. If Palladio, Vignola, and Blondel were not sufficiently critical in their reading of Vitruvius, Lagrange is clearly mistaken in his. For recall that Vitruvius prescribed entasis, not as mere decoration, but as the subtle solution to a difficult engineering problem. Ignorant of this problem, Lagrange abandoned the aesthetic ideal and sought instead a rational basis from which one could judge the value of a given column. With the fanfare: "among those rules at the foundation of architecture there is but one that is fixed and invariable, and consequently susceptible to calculation: that is solidity," Lagrange offered the relative strength of (6).

Though Lagrange cites no source of inspiration for this invective, his contempt for modern architects, his misreading of Vitruvius, and his quest for fixed and unchangeable rules are surely drawn from the ideas of Marc-Antoine Laugier, the anonymous author of the controversial, though very popular, *Essai sur l'Architecture* (1753). Laugier, upset with an architecture that had "been left to the capricious whim of the artists who have offered precepts indiscriminately . . . fixed rules at random, based only on the inspection of ancient buildings, copying the faults as scrupulously as the beauty; lacking principles which would make them see the difference . . . ," summoned the one who "will undertake to save architecture from eccentric opinions by disclosing its fixed and unchangeable laws." [10, p. 2]. For his model Laugier took the primitive hut, a rendering of which served as frontispiece for his work's 2nd edition (1755), see Figure 2. He begins his first chapter with a list, first pronouncing correct methods in the design of columns then remarking on several faulty methods. We recall one of each: "The column must be tapered from bottom to top in imitation of nature where this diminution is found in all plants" [10, p. 14], and "Fault: to give a swelling to the shaft at about the third of its height instead of tapering the column in the normal way. I do not believe that nature has ever produced anything that could justify this swelling" [10, p. 18]. In addition to parroting these opinions Lagrange goes so far as to adopt the vague notion of *solidité*, identified, though undefined, by Laugier as "the first quality a building must have" [10, p. 68]. We shall see that Lagrange, in answering this summons with a cylindrical column, outdoes even Laugier by removing not only the swelling but also the diminution.

As preparation for the general case Lagrange first attacks the finite-dimensional problem of maximizing (6) over those functions of the form

$$A(x) = a + bx + cx^2. \tag{7}$$

Following Euler's lead, Lagrange finds

$$\lambda_c(A) = b^2/4 - ac + \pi^2/h^2, \qquad h \equiv \int_0^\ell \frac{dx}{A},$$

which indeed reduces to (5) when $c = 0$. In the case $b^2 = 4ac$, i.e., $A(x) = (\sqrt{a} + \sqrt{c}x)^2$, he finds

$$\frac{\lambda_c(A)}{V^2(A)} = \frac{\pi^2(a + \sqrt{ac}\,\ell)^2}{\ell^4(a + \sqrt{ac}\,\ell + c\ell^2/3)^2}.$$

For each a this is a decreasing function of c and therefore a maximum when $c = 0$, i.e., among those columns for which A is a perfect square, the cylinder is the strongest. In his subsequent attempt to reduce (7) to a perfect square lies Lagrange's first misstep. In particular, after reducing the relative strength to the workable form that begins his section 24, he errs in setting its logarithmic derivative to zero and therefore arrives at an erroneous necessary condition. This condition implies that perfect square A are indeed to be preferred and hence that the cylinder maximizes (6) over those A given by (7). Offering up this result without physical interpretation Lagrange rushes into the general case of maximizing the relative strength over all functions $A : [0, \ell] \mapsto [0, \infty)$. Again, he finds what he is looking for, the cylinder. The technical errors he was forced to commit at this stage were caught by J. Serret in editing Lagrange's *Oeuvres*.

Had Lagrange had the courage to criticize the physical merits of his scientific design criterion, he would have been led directly to perceive his mistakes of calculus. For maximizing the relative strength is equivalent to maximizing the buckling load subject to fixed volume, and to raise a column's buckling load without changing its volume one should obviously increase A where large bending moment M is expected and decrease it in regions of relatively little bending. In short, A and $|M|$ should be similarly ordered. As the differential equation (4) determines the qualitative properties of the bending moment, this meta-theorem has an immediate consequence. For M, being a, say positive, first eigenfunction of (4), must be a concave function vanishing at each end. Consequently, the buckling load of a hinged cylinder is increased when material is removed from its ends and added to its middle. Finally, nowhere does Lagrange argue the relevance or indicate the role of the chosen hinged boundary conditions in the practical problem he has set himself. He appears to have followed Euler's use of these conditions as blindly as he followed the pronouncements of Laugier.

Though Euler makes no reference to this work of Lagrange, T. Young, arguing that Lagrange possessed "the habit of relying too confidently on calculation, and too little on common sense," believed it "possible to assign a stronger form than a cylinder, since the summit and base must certainly contain some useless matter" [20, p. 568]. T. Clausen in *Über die Form architektonischer Säulen* (1851) [3], was the first to offer a correct solution to this problem of Lagrange. Clausen in fact solved the equivalent problem of minimizing volume subject to a fixed buckling load. I have seen this work only in the summary offered by Pearson [17, v. 2, p. 325], an assessment much clouded by Pearson's ringing endorsement of Lagrange's cylindrical solution. Unaware of Lagrange's historical, physical, and mathematical errors, Pearson credited him with having "shaken the then current architectural fallacies" [17, v. 1, p. 67]. This prattle provoked Truesdell to surmise that "Pearson took [Lagrange] as a torch carrier for Victorian architectural practice, according to which, it seems, the ugliest forms turn out to be the most useful" [6, s. 2, v. 11.2, p. 355]. Confused by a solution which differed from Lagrange's, Pearson endeavored to "simplify" Clausen's analysis. Unfortunately he makes things too simple, for though he arrives at the correct conclusion, the path he takes is nonsense from the start. Rather than dwell on Pearson's mistakes I instead display Clausen's solution, Figure 3 (the exaggerated entasis of a cigar), and move on to J. Keller's derivation in *The shape of the strongest column* (1960) [8].

The Work of J. Keller and I. Tadjbakhsh

Assuming, with respect to the nondimensionalized problem

$$y'' + \lambda A^{-2} y = 0, \quad y(0) = y(1) = 0 \tag{8}$$

$$\int_0^1 A\, dx = V/\ell, \tag{9}$$

that (i) $A \mapsto \lambda_c(A)$ attains its maximum at $\hat{A}$ over those nonnegative A satisfying (9), and (ii) $t \mapsto \lambda_c(\hat{A} + tA_0)$ is differentiable for each variation A_0 satisfying $\int_0^1 A_0\, dx = 0$, Keller succeeded in characterizing $\hat{A}$ via a first-order necessary condition. In particular, the perturbed equilibrium equation

$$y'' + \lambda_c(\hat{A} + tA_0)(\hat{A} + tA_0)^{-2} y = 0,$$
$$y(0) = y(1) = 0,$$

when differentiated with respect to t at $t = 0$, yields

$$\dot{y}'' + \lambda_c(\hat{A})\hat{A}^{-2}\dot{y} = 2\lambda_c(\hat{A})\hat{A}^{-3} A_0 \hat{y},$$
$$\dot{y}(0) = \dot{y}(1) = 0, \tag{10}$$

where I have used the fact that $\dot{\lambda}_c(\hat{A}) = 0$ and denoted the first eigenfunction of (8) when $A = \hat{A}$ by $\hat{y}$. For (10) to possess a solution, the Fredholm alternative requires that its right-hand side be orthogonal to each solution of the corresponding homogeneous equation, i.e., $\int_0^1 \hat{A}^{-3}\hat{y}^2 A_0\, dx = 0$. This being necessary for every zero-mean A_0, there must exist a positive constant c for which

$$\hat{y}^2 = c\hat{A}^3. \tag{11}$$

The form of $\hat{A}$ is now immediate, for in agreement with my simple meta-theorem, like $\hat{y}$ it must be a concave function vanishing at each end. Its precise form is found on solving the nonlinear differential equation, subject to (9), that results on substituting the necessary condition, (11), into the equilibrium equation, (8). Ex-

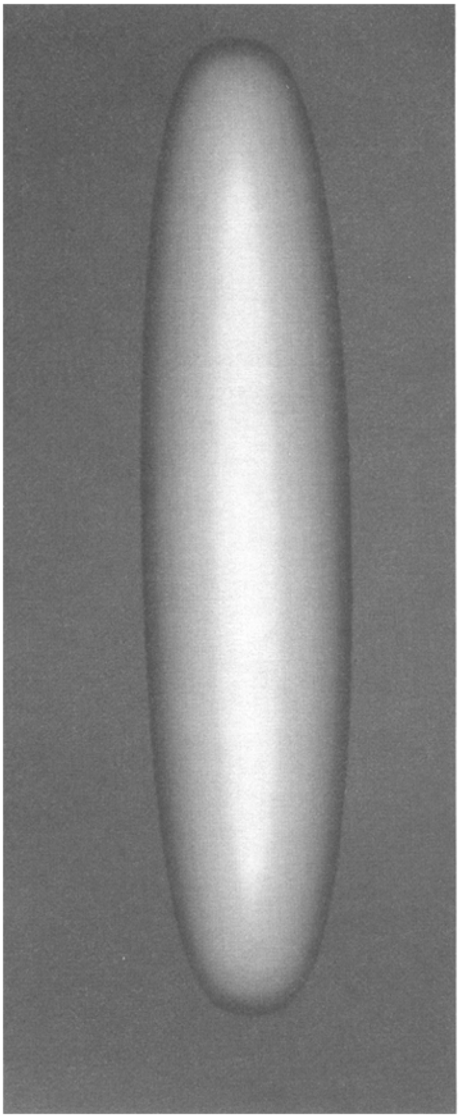

Figure 3. Solution for hinged end conditions.

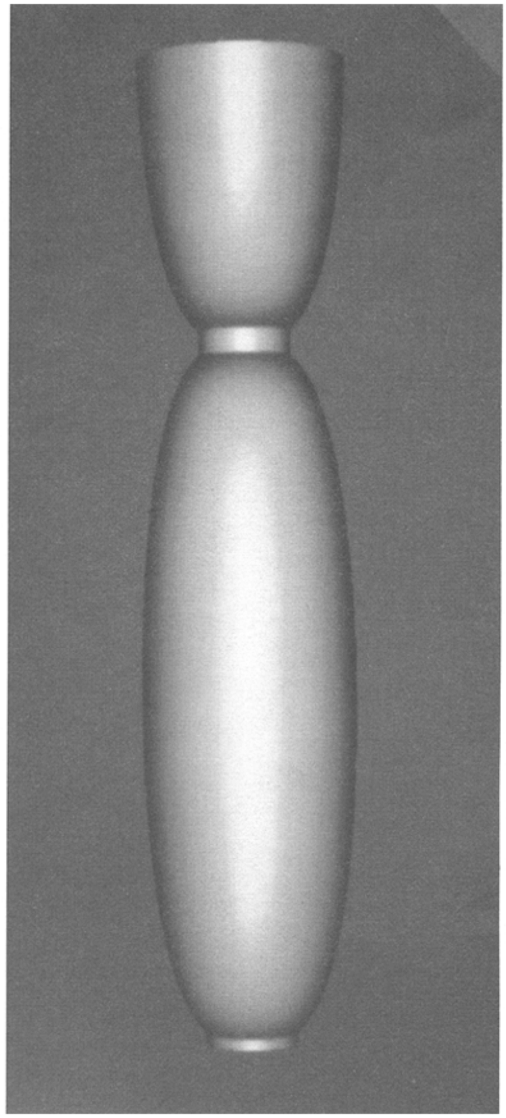

Figure 4. Solution with clamped-hinged end conditions.

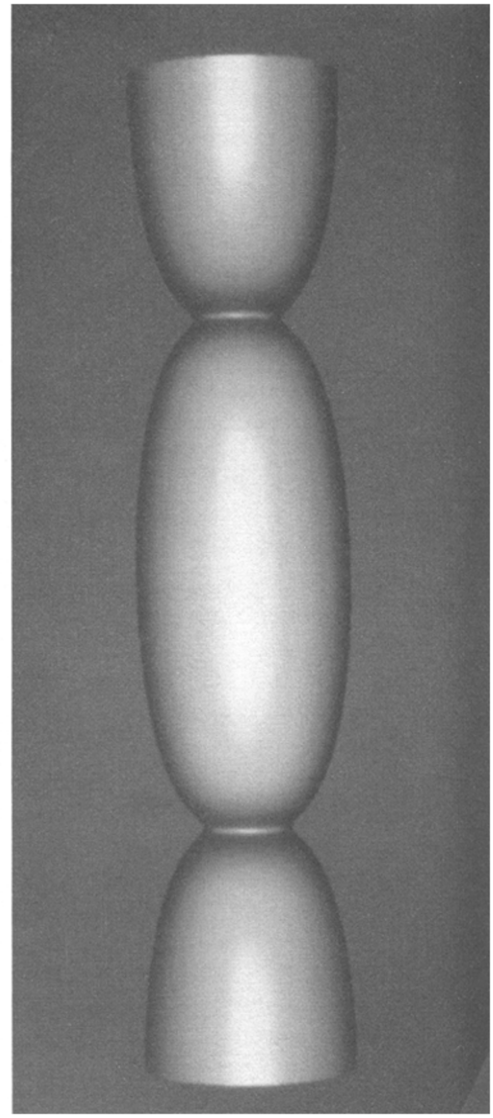

Figure 5. Solution for clamped end conditions.

plicitly, $\lambda_c(\hat{A}) = 4\pi^2V^2/3\ell^2$, while the graph of $\hat{A}$ permits the parametrization

$$\left.\begin{aligned} x(t) &= \frac{3}{4\pi}\left(\frac{2}{3}(t - \sin t)\right) \\ y(t) &= \frac{2}{3}(1 - \cos t) \end{aligned}\right\} \quad 0 \leq t \leq 2\pi.$$

This stunted cycloid, pictured in Figure 3, is stronger, by a factor of 4/3, than the cylindrical column of the same length and volume.

In *Strongest columns and isoperimetric inequalities for eigenvalues* (1962) [16], Keller, with I. Tadjbakhsh, extended his earlier findings to columns either free, hinged, or clamped at their ends. I follow their treatment of the nondimensionalized equilibrium equation

$$(A^2u'')'' + \lambda u'' = 0 \tag{12}$$

subject to either clamped-hinged end conditions,

$$u(0) = u'(0) = 0, \quad u(1) = A^2u''(1) = 0, \tag{13}$$

or clamped end conditions,

$$u(0) = u'(0) = 0, \quad u(1) = u'(1) = 0. \tag{14}$$

Unlike the hinged problem, the displacement u and moment $M = A^2u''$ do not coincide and it is only the moment that satisfies

$$M'' + \lambda A^{-2}M = 0. \tag{15}$$

Assuming existence and smooth dependence, Tadjbakhsh and Keller characterize the optimal design, $\hat{A}$, in terms of its corresponding moment, $\hat{M}$, via $\hat{M}^2 = c\hat{A}^3$ for some positive c. Continuity of $\hat{M}$ then implies continuity of $\hat{A}$, and as $\hat{M} = \hat{A}^2\hat{u}''$ it follows that

$$\hat{A}^4|\hat{u}''|^2 = c\hat{A}^3, \qquad (16)$$

in perfect agreement with (11) when $\hat{y}$ is interpreted as moment. Where (11) led to a design with vanishing cross sectional area at its ends, we shall see that (16) with either (13) or (14) forces $\hat{A}$ to vanish at interior point(s). First note that any nontrivial $C^2(0, 1)$ function obeying (13) admits at least one inflection point, while (14) requires at least two, so in particular, if $\hat{u} \in C^2(0,1)$ then $\hat{u}''(x_0) = 0$ for some $x_0 \in (0, 1)$. Equation (16) then implies that $\hat{A}$ cannot remain bounded in a neighborhood of x_0. As this contradicts the continuity of $\hat{A}$ (not to mention our meta-theorem), one must abandon the assumption that $\hat{u} \in C^2(0, 1)$. The continuity of $\hat{A}$ and (16) then together force $\hat{A}$ to vanish at points where $\hat{u}''$ fails to exist.

Indeed, the designs proposed by Tadjbakhsh and Keller as optimal under clamped-hinged and clamped end conditions possess, respectively, 1 and 2 interior zeros. Fifteen years passed before Olhoff and Rasmussen [12] discovered the buckling load of Tadjbakhsh and Keller's clamped column to be considerably less than advertised. As hard evidence, however, they cited numerical results with no discussion of the algorithm used. Their findings failed to convince those that have argued up through 1988 in favor of Tadjbakhsh and Keller's solution (see the references in [4]). In [4] M. Overton and I established that Olhoff and Rasmussen did however correctly identify the points at which Tadjbakhsh and Keller erred in (i) the calculation of the buckling load of their clamped column, and in (ii) their derivation of the necessary condition (16). At issue in the former is the fact that $\hat{u}'$ need not even exist at points where $\hat{A} = 0$. With this, we found [4, app.] both the clamped-hinged and clamped columns of Tadjbakhsh and Keller to buckle at loads significantly less than the associated cylinders. Regarding (ii), Olhoff and Rasmussen argued, again with supporting numerical data, that unlike second-order problems where eigenvalues can be at most simple, $\lambda_c(A)$ may in fact be double. Under the assumption that $\lambda_c(\hat{A})$ was indeed double for clamped ends, Olhoff and Rasmussen, and later Masur [11] and Seiranian [14], formally derived new necessary conditions. Application of their conditions led, in each case, to the column of Figure 5. In spite of this consensus, doubt remained, for in addition to the formal nature of these derivations, a proof of existence was still lacking. I sketch below the resolution of these two remaining issues.

Getting It Right

In refuting the clamped-hinged and clamped columns of Tadjbakhsh and Keller we found evidence of the not surprising fact that $A \mapsto \lambda_c(A)$ need not be continuous in the sup norm topology over nonnegative functions. This suggests the imposition of a uniform lower bound on those admissable A. In the interest of bounding $\lambda_c(A)$ from above it is convenient to impose, in addition, a uniform upper bound on A. This leaves us with the following set of admissible designs: $ad = \{A \in L^\infty : 0 < \alpha \leq A(x) \leq \beta, \int_0^1 A\,dx = 1\}$. And indeed, for each of the end conditions of interest, there exists an $\hat{A} \in ad$ for which $\lambda_c(\hat{A}) \geq \lambda_c(A)$ for each $A \in ad$ (see [4, §3]). Regarding the differentiability of $A \mapsto \lambda_c(A)$, recall Rayleigh's characterization

$$\lambda_c(A) = \inf \frac{\int_0^1 A^2|u''|^2\,dx}{\int_0^1 |u'|^2\,dx},$$

$$u \in H^2(0,1) \cap ((13) \text{ or } (14)),$$

and denote by $\mathscr{E}(A)$ those (eigen)functions at which this infimum is attained. It is not hard to show that the dimension of $\mathscr{E}(A)$, i.e., the multiplicity of $\lambda_c(A)$, may not exceed two.

Ah, but if the multiplicity is two, that is already enough to invalidate any derivation relying on smooth dependence upon A! As an infimum of smooth functions of A, $A \mapsto \lambda_c(A)$ of course need not be smooth. It is however Lipschitz and therefore amenable to the calculus of Clarke [2]. The generalized gradient of λ_c at $\hat{A}$ is by definition the collection of continuous linear functionals on $L^\infty(0, 1)$ subordinate to the generalized directional derivative of λ_c at $\hat{A}$, i.e., $\partial\lambda_c(\hat{A}) \equiv \{\xi \in (L^\infty)^*; \lambda_c^0(\hat{A}; A) \geq \langle\xi, A\rangle\ \forall A \in L^\infty\}$, where

$$\lambda_c^0(\hat{A}; A) \equiv \limsup_{\substack{B \to \hat{A} \\ t \downarrow 0}} \frac{\lambda_c(B + tA) - \lambda_c(B)}{t}.$$

Where $\partial\lambda_c(\hat{A})$ contains but a single function, λ_c is Gâteaux differentiable and the formal arguments that began with Keller are justified. We found $\partial\lambda_c(\hat{A})$ to be the derivative of the Rayleigh quotient evaluated at its various minimizers. In particular [4, §4],

$$\partial\lambda_c(\hat{A}) = \text{co}\,\{\hat{A}(a\hat{u}_1'' + b\hat{u}_2'')^2 : a^2 + b^2 = 1\},$$

where co denotes convex hull, and $\{\hat{u}_1, \hat{u}_2\}$ spans $\mathscr{E}(\hat{A})$ and obeys $\int_0^1 \hat{u}_i'\hat{u}_j'\,dx = \delta_{ij}$. Zero is not an element of $\partial\lambda_c(\hat{A})$, but rather, for sufficiently large μ, an element of the generalized gradient of the Lagrangian

$$\lambda_c(A) - \mu^2\text{dist}(A,ad),$$

at $\hat{A}$. Consequently, a member of $\partial\lambda_c(\hat{A})$ differs from a positive constant by an amount that is negative when $\hat{A} = \alpha$, positive when $\hat{A} = \beta$, and zero otherwise. More precisely, there exist a $c > 0$ and $\delta_i \geq 0$, $\delta_1\delta_2 \geq \delta_3^2/4$, such that at almost every $x \in (0, 1)$,

$$
\begin{aligned}
\hat{A} = \alpha &\Rightarrow \hat{A}(\delta_1|\hat{u}_1''|^2 + \delta_3\hat{u}_1''\hat{u}_2'' + \delta_2|\hat{u}_2''|^2) \le c \\
\alpha < \hat{A} < \beta &\Rightarrow \hat{A}(\delta_1|\hat{u}_1''|^2 + \delta_3\hat{u}_1''\hat{u}_2'' + \delta_2|\hat{u}_2''|^2) = c \\
\hat{A} = \beta &\Rightarrow \hat{A}(\delta_1|\hat{u}_1''|^2 + \delta_3\hat{u}_1''\hat{u}_2'' + \delta_2|\hat{u}_2''|^2) \ge c
\end{aligned}
\tag{17}
$$

The difference $\delta_1\delta_2 - \delta_3^2/4$ should be interpreted as a Lagrange multiplier that measures interaction between the two buckling modes, $\hat{u}_1$ and $\hat{u}_2$. When this difference is zero, for example, the eigenfunction $\hat{u} \equiv \sqrt{\delta_1}\hat{u}_1 + \sqrt{\delta_2}\hat{u}_2$ satisfies

$$\hat{A}|\hat{u}''|^2 = c, \tag{16}$$

i.e., we recover the necessary condition of Tadjbakhsh and Keller. This clearly occurs when $\lambda_c(\hat{A})$ is simple, and it is not hard to show that this is in fact the case when the end conditions are either hinged or clamped-hinged. Regarding the former, (16) predicts singular behavior of $\hat{u}''$ only at the ends and so Keller's calculation of the buckling load of his hinged column stands. With respect to clamped-hinged conditions, however, recall that (16) forces interior singularities of $\hat{u}''$ and consequent zeros of $\hat{A}$ that invalidate Tadjbakhsh and Keller's calculation of the associated buckling load. Hence (16) in the context of clamped-hinged cannot hold over the column's entire length, i.e., there must exist portions of the column along which $\hat{A}$ is identically α or β. Figure 4 depicts the strongest clamped-hinged column for a particular choice of α and β, obtained numerically in [4].

Though (16) may not hold over the entire column for clamped ends, the same cannot be said for (17). That is, should $\lambda_c(\hat{A})$ be double, so long as $\delta_1\delta_2 - \delta_3^2/4 > 0$ equation (17) in itself does not necessarily require infinite area near zeros of $\hat{u}_i''$ or, conversely, zero area at points where $\hat{u}_i''$ fails to exist. The mixture of the two modes may compensate for the anomalies inherent in any single-mode formulation. Indeed one can choose α sufficiently small and β sufficiently large so that (17) holds over the column's entire length. Figure 5 depicts the strongest clamped column for such a choice, again obtained numerically in [4]. This result vindicates the formal procedures invoked by Olhoff and Rasmussen, Masur, and Seiranian, in deriving the same profile.

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