Statistics Feb 3, 2021

Probability Review

Random variables : X, $\mathcal{U}$

Ex: $X \sim N(\mu, \sigma^2)$

density function f : $\mathbb{P}(X \in A) = \int_A f(x)\,dx$

distribution function: F : $F(x) = \mathbb{P}(X \leq x)$

$$= \int_{-\infty}^{x} f(x)\,dx$$

Expected value : $\mathbb{E}(X) = \int x\,f(x)\,dx$

Variance : $Var(X) = \mathbb{E}((X-\mu)^2)$ $\mu = \mathbb{E}(X)$.

$$= \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

k^{th} moment of X = $\mathbb{E}(X^k)$.

Moment generating function :

$$MGF_X = M_X(t) = \mathbb{E}(e^{tX})$$

$$= \int e^{tx} f(x)\,dx.$$

$$M(0) = \int f(x)\,dx = 1$$

$$M'(t) = \int x\,e^{tx} f(x)\,dx$$

$$M'(0) = \int x\,f(x)\,dx. = \mathbb{E}(X).$$

$$M^{(k)}(0) = \mathbb{E}(X^k).$$

Characteristic Function

$$\varphi_X(t) = \mathbb{E}(e^{itX}) \qquad \cos tx + i \sin tx$$

$$= \int e^{itx} f(x)\, dx$$

$$= \text{Fourier transform of } f$$

$$= \hat{f}(t).$$

Transformations

X has density f, let $Y = g(X)$.

What is the density function for Y?

(assume g is monotone increasing)

$$\mathbb{P}(Y \in A) = \mathbb{P}(g(X) \in A)$$

$$= \mathbb{P}(X \in g^{-1}(A)).$$

$$= \int_{g^{-1}(A)} f(x)\, dx$$

$$= \int_A \underbrace{f(g^{-1}(y)) \frac{dg^{-1}}{dx}}_{h(y),\ \text{the density of } Y.} dy$$

let $x = g^{-1}(y)$

$\Rightarrow g(x) = y \qquad y \in A$

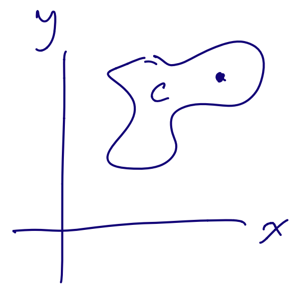
$\frac{dg}{dx}\, dx = dy$

$dx = \underbrace{\frac{1}{dg/dx}} dy$

Multi-dimensional distributions.

Consider 2 random variables X, Y.

Joint probability density: $f(x,y)$



$$\mathbb{P}(X \in A, Y \in B) = \int_B \int_A f(x,y)\, dx\, dy$$

$$\mathbb{P}(X,Y \in C) = \int_C f(x,y)\, dx\, dy$$

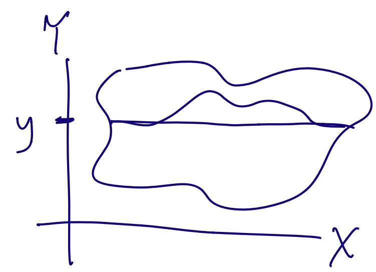
X, Y are independent if $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\, \mathbb{P}(X \in B)$.

$$\Rightarrow \quad f(x,y) = \underbrace{f_X(x)}\ \underbrace{f_Y(y)}.$$

marginal densities

$$f_X(x) = \int f(x,y)\, dy.$$

Conditional random variables



$$\mathbb{P}(X \in A \mid Y = y) = \frac{f(x,y)}{f_Y(y)}. = f_{X|Y}(x|y).$$

$$\Rightarrow \quad \int \frac{f(x,y)}{f_Y(y)}\, dx = \frac{1}{f_Y(y)} \underbrace{\int f(x,y)\, dx}$$

$$= 1.$$

Bivariate transformations

$U = g_1(X, Y)$
$V = g_2(X, Y)$ $\Rightarrow$ What is the density of U, V?

Covariance and correlation

$Cov(X, Y) = \mathbb{E}((X-\mu_X)(Y-\mu_Y)) = Cov(Y, X).$

If X, Y are independent, then:

$$= \mathbb{E}(X-\mu_X)\,\mathbb{E}(Y-\mu_Y)$$

$$= 0.$$

Correlation of X, Y $= \rho_{XY} = \dfrac{Cov(X,Y)}{\underbrace{\sqrt{Var(X)}}_{\text{Standard deviation of } X}\sqrt{Var(Y)}} \in [-1, 1].$

Multidimensional Normal Random variables

$X_1, \dots, X_n$ be $N(\mu_i, \sigma_{ii}^2)$ random variables.

and $Cov(X_i, X_j) = \sigma_{ij}^2$

The $\underbrace{\text{covariance matrix}}_{n\times n \text{ matrix}} = C_{ij} = \sigma_{ij}^2$

$$C = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \sigma_{13}^2 & \cdots & \sigma_{1n}^2 \\ \vdots & \sigma_{22}^2 & & & \\ \vdots & & \ddots & & \\ & & & \ddots & \\ \sigma_{n1}^2 & & & & \end{pmatrix}$$

joint pdf $= \dfrac{1}{(2\pi)^{n/2}} \dfrac{1}{\sqrt{\underbrace{|C|}_{\text{determinant of } C}}} e^{(\vec{x}-\mu)^T C^{-1} (\vec{x}-\mu)/2}$ $\qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$

$= f(x_1, \dots, x_n).$

$$\int f(x_1,\dots,x_n)\, dx_1 \dots dx_n = 1.$$

$$\mathbb{E}[X_j] = \int x_j\, f(x_1 \dots x_n)\, dx_1 \dots dx_n.$$

Inequalities

Markov inequality:

If $X > 0$, and $\mathbb{E}(X) < \infty$, then

for any $t > 0$,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}$$

$$\mathbb{E}(X) = \int_0^\infty x f(x)\, dx = \int_0^t x\, f(x)\, dx + \int_t^\infty x\, f(x)\, dx$$

$$\geq \int_t^\infty x\, f(x)\, dx$$

$$\geq t \int_t^\infty f(x)\, dx = t\, \mathbb{P}(X > t)$$

$$\Rightarrow \mathbb{E}(X) \geq t\, \mathbb{P}(X > t).$$

$$\Rightarrow \mathbb{P}(X > t) \leq \frac{\mathbb{E}(X)}{t}.$$