Homework 6

Due: 2:00pm Mar. 10rd, 2016

Each problem is worth 10 points.

Exercise 1 [Piecewise spline]: Determine the piecewise polynomial functions $p_1(x)$ and $p_2(x)$ which define the function $p(x)$:

$$p(x) = \begin{cases} p_1(x) & \text{for } 0 \leq x \leq 1, \\ p_2(x) & \text{for } 1 \leq x \leq 2, \end{cases}$$

such that:

- $p_1(x)$ is linear,
- $p_2(x)$ is quadratic,
- $p(x)$ and $p'(x)$ are continuous at $x = 1$,
- $p(0) = 1$, $p(1) = -1$, and $p(2) = 0$.

Exercise 2 [Approximation error]: Using Taylor's Theorem, derive the error term for the approximation:

$$f'(x) \approx \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}.$$

What is the round-off error in the above finite difference? (You can ignore the error in computing x, $x + h$, and $x + 2h$.

Exercise 3 [Optimal forward difference]: Consider the following finite difference expression for f'':

$$f''(x) \approx Af(x) + Bf(x+h) + Cf(x+2h).$$

Use Taylor's Theorem to determine A, B, and C that give the maximal order of accuracy, and determine what this order is.

Exercise 4 [Spectral differentiation]: The Chebyshev polynomials have indefinite integrals given by:

$$\int T_n(x)\, dx = \frac{1}{2}\left(\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right) + C, \qquad n = 2, 3, \ldots,$$

where C is an arbitrary constant. Indefinite integrals of T_0 and T_1 can be computed directly.

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(a) Suppose that

$$p(x) = \sum_{n=0}^{N} a_n \, T_n(x).$$

Determine coefficients $A_0, \ldots, A_{N+1}$ such that

$$\int p(x) \, dx = \sum_{n=0}^{N+1} A_n \, T_n(x),$$

e.g., express $A_0, \ldots, A_{n+1}$ in terms of $a_0, \ldots, a_n$. The coefficient A_0 can be arbitrary to account for the arbitrary constant of integration.

(b) Now suppose that

$$q(x) = \sum_{n=0}^{N+1} A_n \, T_n(x).$$

Reverse the process in part (a) to determine coefficients $a_0, \ldots, a_n$ (in terms of $A_0, \ldots A_{N+1}$) such that

$$q'(x) = \sum_{n=0}^{N} a_n \, T_n(x).$$