1) Consider the linear heat equation \( u_t - u_{xx} = 0 \) on the interval \( 0 < x < 1 \), with boundary condition \( u = 0 \) at \( x = 0,1 \) and initial condition \( u = 1 \).

   (a) Interpret \( u \) as the value of a suitable double-barrier option.

   (b) Express \( u(t, x) \) as a Fourier sine series, as explained in Section 3.

   (c) At time \( t = 1/100 \), how many terms of the series are required to give \( u(t, x) \) within one percent accuracy?

2) Consider the SDE \( dy = f(y)dt + g(y)dw \). Let \( G(x,y,t) \) be the fundamental solution of the forward Kolmogorov PDE, i.e. the probability that a walker starting at \( x \) at time 0 is at \( y \) at time \( t \). Show that if the infinitesimal generator is self-adjoint, i.e.

   \[-(fu)_x + \frac{1}{2}(g^2u)_{xx} = fu_x + \frac{1}{2}g^2u_{xx},\]

then the fundamental solution is symmetric, i.e. \( G(x,y,t) = G(y,x,t) \).

3) Consider the stochastic differential equation \( dy = f(y,s)ds + g(y,s)dw \), and the associated backward and forward Kolmogorov equations

   \[ u_t + f(x,t)u_x + \frac{1}{2}g^2(x,t)u_{xx} = 0 \quad \text{for } t < T, \text{ with } u = \Phi \text{ at } t = T \]

   and

   \[ \rho_s + (f(z,s)\rho)_z - \frac{1}{2}(g^2(z,s)\rho)_{zz} = 0 \quad \text{for } s > 0, \text{ with } \rho(z) = \rho_0(z) \text{ at } s = 0. \]

Recall that \( u(x,t) \) is the expected value (starting from \( x \) at time \( t \)) of payoff \( \Phi(y(T)) \), whereas \( \rho(z,s) \) is the probability distribution of the diffusing state \( y(s) \) (if the initial distribution is \( \rho_0 \)).

   (a) The solution of the backward equation has the following property: if \( m = \min_z \Phi(z) \) and \( M = \max_z \Phi(z) \) then \( m \leq u(x,t) \leq M \) for all \( t < T \). Give two distinct justifications: one using the maximum principle for the PDE, the other using the probabilistic interpretation.

   (b) The solution of the forward equation does not in general have the same property; in particular, \( \max_z \rho(z,s) \) can be larger than the maximum of \( \rho_0 \). Explain why not, by considering the example \( dy = -yds \). (Intuition: \( y(s) \) moves toward the origin; in fact, \( y(s) = e^{-s}y_0 \). Viewing \( y(s) \) as the position of a moving particle, we see that
particles tend to collect at the origin no matter where they start. So \( \rho(z, s) \) should be increasingly concentrated at \( z = 0 \).) Show that the solution in this case is \( \rho(z, s) = e^s \rho_0(e^s z) \). This counterexample has \( g = 0 \); can you also give a counterexample using \( dy = -y ds + \epsilon dw \)?

4) On planet Dough a risky asset \( S_t \) evolves according to the diffusion equation

\[
dS_t = a(S_t) \, dt + b(S_t) \, dW_t
\]

while a risk-free asset \( X_t \) evolves according to

\[
dx_t = c(X_t) \, dt.
\]

Here \( (a, b, c) \) are arbitrary functions, which generalize the usual log-normal model in which we had \( a = \mu S_t \), \( b = \sigma S_t \), and \( c = r X_t \). Adapt the derivation of the Black–Scholes equation to this more general case and find the PDE that determines the fair price of a derivative \( V_t = v(S_t, t) \) on this planet. Is the drift term \( a(S_t) \) important on this planet?

5) Consider the solution of

\[
u_t + a u_{xx} = 0 \quad \text{for} \quad t < T, \quad \text{with} \quad u = \Phi \quad \text{at} \quad t = T
\]

where \( a \) is a positive constant. Recall that in the stochastic interpretation, \( a \) is \( \frac{1}{2} g^2 \) where \( g \) represents volatility. Let’s use the maximum principle to understand qualitatively how the solution depends on volatility.

(a) Show that if \( \Phi_{xx} \geq 0 \) for all \( x \) then \( u_{xx} \geq 0 \) for all \( x \) and \( t \). (Hint: differentiate the PDE.)

(b) Suppose \( \bar{u} \) solves the analogous equation with \( a \) replaced by \( \bar{a} > a \), using the same final-time data \( \Phi \). We continue to assume that \( \Phi_{xx} \geq 0 \). Show that \( \bar{u} \geq u \) for all \( x \) and \( t \). (Hint: \( w = \bar{u} - u \) solves \( w_t + \bar{a} w_{xx} = f \) with \( f = (a - \bar{a}) u_{xx} \leq 0 \).)

6) Consider the standard finite difference scheme

\[
\frac{u((m + 1)\Delta t, n\Delta x) - u(m\Delta t, n\Delta x)}{\Delta t} = \frac{u(m\Delta t, (n + 1)\Delta x) - 2u(m\Delta t, n\Delta x) + u(m\Delta t, (n - 1)\Delta x)}{(\Delta x)^2}
\]

for solving \( u_t - u_{xx} = 0 \). The stability restriction \( \Delta t < \frac{1}{2} \Delta x^2 \) leaves a lot of freedom in the choice of \( \Delta x \) and \( \Delta t \). Show that

\[
\Delta t = \frac{1}{6} \Delta x^2
\]

is special, in the sense that the numerical scheme has errors of order \( \Delta x^4 \) rather than \( \Delta x^2 \). In other words: when \( u \) is the exact solution of the PDE, the left and right sides of (3) differ by a term of order \( \Delta x^4 \). [Comment: the argument sketched in the Section 3 Addendum shows that if \( u \) solves the PDE and \( v \) solves the finite difference scheme then \( |u - v| \) is of order \( \Delta x^2 \) in general, but it is smaller – of order \( \Delta x^4 \) – when \( \Delta t = \frac{1}{6} \Delta x^2 \).]