Recall the Fourier transform pair
\[ \hat{f}(k) = \int_{-\infty}^{+\infty} f(x) \exp(-ikx) \, dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(k) \exp(+ikx) \, dk \]
and the basic Fourier representation of the delta function
\[ \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(ikx) \, dk. \]

1. **Convolution rule**

Given \( f(x) \) and \( g(x) \) the convolution operator \( * \) is defined as
\[ h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(x-s)g(s) \, ds. \]

Show from the definition that the convolution is symmetric (i.e., \( f * g = g * f \)) and that
\[ \hat{h}(k) = \hat{f}(k) \hat{g}(k). \]

2. **Fourier transform and regularity**

Consider the function \( l > 0 \) is some constant
\[ f(x) = \begin{cases} 
0 & x < 0 \\
x & 0 < x < l \\
2l-x & l < x < 2l \\
0 & 2l < x 
\end{cases} \]

Based on the regularity of \( f(x) \), what power-law decay for large \( |k| \) do you expect for \( \hat{f}(k) \)?

Compute \( \hat{f}(k) \) and verify your expectation. Repeat these two steps for the derivatives \( g(x) = f'(x) \) and \( h(x) = g'(x) \). How is \( \hat{g}(k) \) related to the Fourier transform of a unit block of width \( l \) computed in class?

3. **Stationary phase for group velocity and dispersive caustic**

Recall the integral
\[ u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_0(k) \exp(i[kx - \omega(k)t]) \, dk \]
for a dispersive wave problem with dispersion function \( \omega(k) \) and initial conditions \( u(x,0) = u_0(x) \). As in class, for compactly supported and initial data at the origin, seek an asymptotic evaluation of \( u \) along a ray with fixed \( c = x/t \) for large \( t = 1/\epsilon \). This brings in the wavenumber \( k_0 \) such that \( \omega'(k_0) = c \). Find the leading-order term of the solution if \( \omega''(k_0) = 0 \) and \( \omega'''(k_0) \neq 0 \) (you do not need to compute the numerical value of this term exactly, but you need to find its dependence on \( t = 1/\epsilon \) and \( \omega'''(k_0) \)). This is an example of a dispersive caustic. For large enough \( t \), does the solution along a caustic ray decay faster or slower than on a non-caustic case?

Where in this group-velocity derivation (caustic or not) did you use the assumption that \( u_0(x) \) is compact? Discuss how one could relax compactness to a decay condition as \( |x| \to \infty \). Hint: find a bound on the size of \( d\hat{u}_0(k)/dk \) in terms of a certain integral over \( u_0(x) \).