1. Refraction of sound in two dimensions

Consider the ray-tracing frequency function for sound propagation in two dimensions:
\[
\Omega(k, l, x) = c(x) \sqrt{k^2 + l^2}.
\]
Here \(c(x)\) denotes the wave speed, which depends on \(x\). Write down the ray-tracing equations for \(\{x(t), y(t), k(t), l(t)\}\) and show that \(l\) and \(\omega\) are constant along a ray. If a ray starts with
\[
x(0) = 0, \quad y(0) = 0, \quad k(0) = 0, \quad l(0) = 1
\]
then show that the ray slope along the trajectory is
\[
\frac{dy}{dx} = \frac{1}{\sqrt{\frac{c_0^2}{c^2} - 1}}
\]
where \(c_0 = c(0)\). Hint: use the constancy of \(l\) and \(\omega\) along the ray. In the special case of
\[
\frac{c(x)}{c_0} = \frac{1}{\sqrt{1 + x}}
\]
find the ray parametrized as \(y(x)\) and sketch it in the \(xy\)-plane.

2. Hilbert transform

Let \(f(x)\) be a complex-valued function of \(x\) on the real line with Fourier transform \(\hat{f}(k)\). Then the Hilbert transform of \(f(x)\) is the complex-valued function \(F(x)\) defined by its Fourier transform as
\[
\hat{F}(k) = -i \text{sgn}(k) \hat{f}(k).
\]
Hence both \(f(x)\) and its Hilbert transform \(F(x)\) are functions of \(x\). Based on this definition and what you know about Fourier transforms, verify the following properties of \(F(x)\).

(a) The Hilbert transform of a Hilbert transform brings back the original function \(f(x)\) with a minus sign.

(b) The quadratic norm satisfies
\[
\int_{-\infty}^{+\infty} |f|^2 \, dx = \int_{-\infty}^{+\infty} |F|^2 \, dx.
\]

(c) If \(f(x)\) is a real function then so is \(F(x)\).

(d) The Hilbert transform of \(\cos x\) is \(\sin x\) and that of \(\sin x\) is \(-\cos x\).

(e) The Hilbert transform has the equivalent integral expression
\[
F(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(s)}{x - s} \, ds
\]
where the integral is defined by its principal value at the singularity \(x = s\) (you don’t need to show this). Use this integral to compute the Hilbert transform of the unit block function \(f = 0\) if \(x < 0\) or \(x > 1\), and \(f = 1\) otherwise. This demonstrates that the Hilbert transform of a discontinuous function is typically unbounded; is this consistent with (6)?