Recall the Fourier transform pair

\[ \hat{f}(k) = \int_{-\infty}^{+\infty} f(x) \exp(-ikx) \, dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(k) \exp(+ikx) \, dk \]  

(1)

and the basic Fourier representation of the delta function

\[ \int_{-\infty}^{+\infty} \exp(ikx) \, dk = 2\pi \delta(x). \]  

(2)

1. Convolution rule

Given \( f(x) \) and \( g(x) \) the convolution operator \(*\) is defined as

\[ h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(x-s)g(s) \, ds. \]  

(3)

Show from the definition that the convolution is symmetric (i.e., \( f * g = g * f \)) and that

\[ \hat{h}(k) = \hat{f}(k) \hat{g}(k). \]  

(4)

2. Fourier transform and regularity

Consider the function \((l > 0 \text{ is some constant})\)

\[ f(x) = \begin{cases} 
0 & x < 0 \\
x & 0 < x < l \\
2l - x & l < x < 2l \\
0 & 2l < x
\end{cases}. \]  

(5)

Based on the regularity of \( f(x) \), what power-law decay for large \(|k|\) do you expect for \( \hat{f}(k) \)? Compute \( \hat{f}(k) \) and verify your expectation. Repeat these two steps for the derivatives \( g(x) = f'(x) \) and \( h(x) = g'(x) \). How is \( \hat{g}(k) \) related to the Fourier transform of a unit block of width \( l \) computed in class?

3. Stationary phase for group velocity and dispersive caustic

Recall the integral

\[ u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}_0(k) \exp(i[kx - \omega(k)t]) \, dk \]  

(6)

for a dispersive wave problem with dispersion function \( \omega(k) \) and initial conditions \( u(x,0) = u_0(x) \). As in class, for compactly supported initial data at the origin, seek an asymptotic evaluation of \( u \) along a ray with fixed \( c = x/t \) for large \( t = 1/\epsilon \). This brings in the wavenumber \( k_0 \) such that \( \omega'(k_0) = c \). Find the leading-order term of the solution if \( \omega''(k_0) = 0 \) and \( \omega'''(k_0) \neq 0 \) (you do not need to compute the numerical value of this term exactly, but you need to find its dependence on \( t = 1/\epsilon \) and \( \omega'''(k_0) \)). This is an example of a dispersive caustic. Does the solution along the caustic ray decay faster or slower than in the non-caustic case \( \omega''(k_0) \neq 0 \)?

Where in this group-velocity derivation (caustic or not) did you use the assumption that \( u_0(x) \) is compact? Hint: consider the size of

\[ \frac{d\hat{u}_0}{dk} = \int_{-\infty}^{+\infty} -ix \, u_0(x) \exp(-ikx) \, dx. \]  

(7)