1. Matched asymptotic expansions

Consider the boundary-value problem for $y(x)$ as studied in class:

$$\epsilon y'' + y' + y = 0 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y(1) = 1/e.$$  \hspace{1cm} (1)

Using $X = x/\epsilon$ pose two-term inner and outer expansions of the form

$$Y(X, \epsilon) \sim Y_0(X) + \epsilon Y_1(X) \quad \text{and} \quad y(x, \epsilon) \sim y_0(x) + \epsilon y_1(x)$$  \hspace{1cm} (2)

and solve for the four unknown functions $(Y_0, Y_1, y_0, y_1)$ up to two undetermined constants that arise in the inner solution. Hint: the first-order inner solution $Y_1$ should contain a secular term proportional to $X$.

The constants are to be determined by matching the expansions via an intermediate variable $\eta = x/\epsilon^\alpha$ such that

$$x = \eta \epsilon^\alpha \quad \text{and} \quad X = \eta^\alpha - 1.$$  \hspace{1cm} (3)

The a priori range of $\alpha$ is $0 < \alpha < 1$ in order to assure that $X \to \infty$ and $x \to 0$ as $\epsilon \to 0$ at fixed $\eta$. However this range is further narrowed by the requirement that the inner expansion remains asymptotic as this limit is taken and $X$ grows without bound. (There is no problem with the outer expansion because $x \in [0, 1]$ has a finite range.) Of course, this all turns on secular terms in the expansion.

To narrow the range of $\alpha$ extend the inner expansion by a second-order term $\epsilon^2 Y_2(X)$; you don’t need to compute the details of $Y_2$ but show that it will contain a term proportional to $X^2$. Hence show that for $\epsilon^2 Y_2$ to remain small compared to all terms in $\epsilon Y_1$ as the limit $\epsilon \to 0$ at fixed $\eta$ is taken, we must restrict the range of $\alpha$ to be

$$\frac{1}{2} < \alpha < 1.$$  \hspace{1cm} (4)

With this result in hand match the expansion to find the two undetermined constants. Hint: there will be terms $\exp(-\eta \epsilon^\alpha)$ that go to zero exponentially fast so that they can be ignored and there will be terms $\exp(-\eta \epsilon^\alpha)$ that should be expanded in a Taylor series.

Form the composite expansion by adding the inner and outer expansion and subtracting the overlap terms, which are for instance given by the outer limit of the inner solution as $X \to \infty$ (hint: the overlap terms should be $1 + \epsilon - x$). State your final answer for the composite expansion as a function of $x$ and $\epsilon$.

Find the exact solution to (1) and recall from class that the one-term composite expansion for this problem is

$$y \sim \exp(-x) - \exp(-x/\epsilon).$$  \hspace{1cm} (5)

Use matlab to plot the exact solution and the one-term and two-term composite expansions for three different values of $\epsilon$, namely $(1/3, 1/5, 1/10)$. Comment on the accuracy of either of the two asymptotic solutions in the three cases.