1. Dimensional analysis

The drag force $F$ on a car of a given shape is assumed to be a function of the velocity $U$, the car height $H$, the air density $\rho$, and the kinematic viscosity of air $\nu$. Use dimensional analysis to find an expression for the non-dimensional drag (there are two equivalent ways of doing this). What is the condition for dynamic similarity in this example?

The real case of interest has parameters $U = 108 \text{ km/h}$ and $H = 1.5 \text{ m}$. In a wind tunnel experiment only a model with $\tilde{H} = 1 \text{ m}$ will fit. What wind velocity $\tilde{U}$ must be chosen to make the experiment dynamically similar to the real case? How is the measured drag force $\tilde{F}$ related to the real $F$? In the experiment an unwanted oscillation with frequency $70 \text{ Hz}$ is measured, what frequency would you expect in the real car?

2. Similarity

Let $u(x, t)$ solve the initial-value problem

$$u_t = u_{xxx} \quad \text{with} \quad u(x, 0) = f(x). \quad (1)$$

Let $v(x, t)$ solve the stretched version

$$v_t = v_{xxx} \quad \text{with} \quad v(x, 0) = f(\alpha x) \quad (2)$$

where $\alpha > 0$ is a constant. Find a rescaling of the independent variables $(x, t)$ that allows expressing $v$ in terms of $u$. Give an explicit formula of $v(x, t)$ in terms of $u$.

3. WKB

Use the WKB method from the lectures to find the general form of the WKB solution for $\phi(t)$ governed by

$$\phi''' - \Omega^4 \phi = 0 \quad (3)$$

where $\Omega(\epsilon t) > 0$ with $\epsilon \ll 1$ is a slowly varying frequency function. Is there an analogue of the action conservation law?

4. Stationary phase

Recall the integral

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}_0(k) \exp(i[kx - \omega(k)t]) \, dk \quad (4)$$

for a dispersive wave problem with dispersion function $\omega(k)$. As in class, seek an asymptotic evaluation of $u$ along a ray with fixed $c = x/t$ for large $t = 1/\epsilon$. This brings in the wavenumber $k_0$ such that $\omega'(k_0) = c$. Find the leading-order term of the solution if $\omega''(k_0) = 0$ and $\omega'''(k_0) \neq 0$. This is an example of a dispersive caustic. Does the solution along the ray decay faster or slower than in the case $\omega''(k_0) \neq 0$?

(please turn over)
5. Multiple scales

Consider the following initial-value problem for $\phi(t)$:

$$
\phi'' + \phi = \epsilon \phi^3 \quad \text{with} \quad \phi(0) = 0, \quad \phi'(0) = 1.
$$

(5)

Using a slow time scale $\tau = \epsilon t$ and the asymptotic expansion $\phi(t, \tau, \epsilon) \sim \phi_0(t, \tau) + \epsilon \phi_1(t, \tau)$ find the leading-order solution valid for $\epsilon t = O(1)$.

Hint: it’s useful to write the general solution of the linear harmonic oscillator equation in $t$ as $A(\tau) \sin(t + \theta(\tau))$ where $A(\tau)$ is a slowly varying amplitude and $\theta(\tau)$ is a slowly varying phase shift.

Using careful scaling, apply your solution to the nonlinear pendulum equations with small initial conditions:

$$
\phi'' + \sin \phi = 0 \quad \text{with} \quad \phi(0) = 0, \quad \phi'(0) = \epsilon.
$$

(6)

6. Kapitza’s horizontal pendulum

Adapt the multi-scale computation given in class to the case of a nonlinear pendulum whose point of support is oscillating horizontally. The scaled governing equations are

$$
\phi'' + \sin \phi + \frac{\beta}{\epsilon} \cos(t/\epsilon) \cos \phi = 0.
$$

(7)

Find the effective potential energy $V(\phi_0)$ for the leading-order solution and plot it for different values of $\beta$. Where are the stable fixed points as a function of $\beta$?

7. Matched asymptotic expansions

Reconsider the boundary-value problem for $y(x)$ studied in class:

$$
\epsilon y'' + y' + y = 0 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y(1) = 1.
$$

(8)

Using $X = x/\epsilon$ pose two-term inner and outer expansions of the form

$$
Y(X, \epsilon) \sim Y_0(X) + \epsilon Y_1(X) \quad \text{and} \quad y(x, \epsilon) \sim y_0(x) + \epsilon y_1(x)
$$

(9)

and solve for the four unknown functions up to two undetermined constants. Determine the constants using matching via the intermediate variable $\eta = x/\sqrt{\epsilon}$. Plot the composite expansion against the exact solution using matlab.