European Journal of Mechanics B/Fluids 47 (2014) 12–15

Contents lists available at ScienceDirect

European Journal of Mechanics B/Fluids

journal homepage: www.elsevier.com/locate/ejmflu

A gentle stroll through EP flux theory

Oliver Bühler *

Center for Atmosphere Ocean Science and Department of Mathematics, Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA

ARTICLE INFO

Article history:
Available online 7 February 2014

Keywords:
Wave activity flux
Pseudomomentum
Non-accelereation conditions

ABSTRACT

The celebrated 1960 paper by Eliassen & Palm (hereafter EP) put on record several brilliant discoveries in the theory of linear waves on shear flows for rotating stratified fluid systems. These discoveries opened up a new perspective on linear wave dynamics in the atmosphere and on the nascent theory of nonlinear interactions between the waves and the mean flow. Arguably, the most important discovery was that of their eponymous wave activity flux vector in the meridional plane and of the conditions under which this important flux was non-divergent.

In this short paper we will retrace some of the steps of EP and explore how their path-breaking discoveries came to be understood in the light of subsequent theories. Of course, an endeavour like this runs the risk of looking patronizing, if only because of 50 years of hindsight, but this is not intended: it was the power of their original discoveries that inspired five decades of further research, with new results still coming out today.

1. A beautiful flux

In their famous[1] paper [1], EP analysed in detail two equation sets for linear stationary waves: first a non-rotating two-dimensional vertical slice model with an arbitrary basic zonal flow $U(z)$, and second a rotating three-dimensional model with a basic zonal flow $U(y, z)$ and a corresponding basic buoyancy field $B(y, z)$ in thermal wind balance. Here stationary means that the waves are time-independent in the chosen reference frame, i.e., the absolute frequency of the waves is zero. Their attention was focused on the dynamics of *wave energy*, which in a Boussinesq setting[2] has the simple density

$$E = \frac{1}{2}\left(u^2 + v^2 + w^2 + \frac{b^2}{N^2}\right). \tag{1}$$

Here (u, v, w) are the velocity components of the linear wave field, b is the linear buoyancy disturbance and $N^2 = B_z$ is the buoyancy frequency. The wave energy is the obvious intrinsic measure of wave activity relative to a variable mean flow U and it therefore came as a considerable surprise that wave energy is not conserved unless U is constant. More precisely, when the wave energy budget is formulated, there are flux terms as well as source or sink terms to do with the mean shear. But [1] achieved far more than merely pointing out this surprising fact!

By manipulations that appear as dreamlike and mysterious now as they did 50 years ago, they derived a non-divergent flux vector field $\boldsymbol{F}$ in the meridional plane that is now called the Eliassen–Palm flux:

$$\boldsymbol{F} = \hat{\boldsymbol{y}}\left(-\overline{uv} + \overline{bv}\frac{U_z}{N^2}\right) + \hat{\boldsymbol{z}}\left(-\overline{uw} + \overline{bv}\frac{f - U_y}{N^2}\right)$$
$$\Rightarrow \nabla\cdot\boldsymbol{F} = 0. \tag{2}$$

This is their Eq. (10.8). Here $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ are unit vectors in latitude and altitude, the overbar denotes zonal averaging, and $f(y)$ is the Coriolis parameter, which may depend on the latitude y. The non-divergence property holds to second order in wave amplitude under the stated assumptions of stationary waves without dissipation and without critical layers. The generality of the EP flux result is remarkable: it holds for arbitrary $f(y)$ and for arbitrary zonal flows $U(y, z)$ and corresponding buoyancy fields $B(y, z)$ in thermal wind balance. Indeed, the thermal wind balance $fU_z = -B_y$ is needed to derive (2), a reminder of the fact that in general the basic flow must be a solution of the governing equations in order to get correct linear disturbance results.

In time, $\boldsymbol{F}$ came to be recognized as an excellent diagnostic tool with which to analyse the propagation of waves in the meridional

* Tel.: +1 2129983265.
E-mail address: obuhler@cims.nyu.edu.

[1] The paper's year of publication is commonly cited as 1961, but the paper apparently appeared already in 1960.

[2] Eliassen and Palm did not restrict themselves to the Boussinesq equations, but all their main points apply more easily in that setting.

http://dx.doi.org/10.1016/j.euromechflu.2014.01.010
0997-7546/

plane [2]. But the EP flux also proved to be extremely important for the theory of nonlinear wave–mean flow interactions in a rotating stratified fluid. Both of these aspects of $\boldsymbol{F}$ were fully understood only a long time after the publication of their paper, basically once it became clear that $\boldsymbol{F}$ had little to do with the flux of wave energy, but was instead found to be equal to (minus) the flux of wave *pseudomomentum*. Let us now look a little closer at EP's discoveries about the energy budget and the subsequent research they inspired!

2. In search of lost wave energy

In the non-rotating two-dimensional vertical slice model the velocity field lies entirely in the xz-plane and nothing depends on y. Any shear flow $U(z)$ is a trivial steady state of the system and EP considered linear stationary waves relative to such a shear flow under the restriction that $U(z) > 0$, say, which rules out critical layers. Under these assumptions EP derived two important results about the mean vertical momentum flux $\overline{uw}$ and the mean vertical wave energy flux $\overline{pw}$ (here p is the linear pressure disturbance):

$$(\overline{uw})_z = 0 \quad \text{and} \quad \overline{pw} = -U\,\overline{uw}. \tag{3}$$

Both results are remarkable. The first one is immediately relevant because in this simple system the zonal mean flow acceleration is obviously given by $-(\overline{uw})_z$, which is the divergence of the standard "Reynolds stress". EP's first result then makes obvious that under the stated assumptions the mean flow is in fact accelerated nowhere at all! Statements of this kind have since become known as "non-acceleration conditions", with significant further contributions already coming out in [3], who referred to EP's work.

The second result shows that, in contrast, the vertical wave energy flux is not constant in z, but varies in magnitude in proportion with the mean flow $U(z)$. For example, for an upward propagating lee wave with negative momentum flux $\overline{uw} < 0$ this means that wave energy is somehow created if $U_z > 0$, say. Total energy is conserved, so somehow this net gain of wave energy must be compensated by a corresponding net loss of mean flow energy. This is very confusing and mysterious, as the first result showed that the mean flow does not change anywhere!

Arguably, this energy budget puzzle can only be resolved by considering the time-dependent spin-up of the wave field (e.g., [4]). In such a scenario the linear wave fields are allowed to depend on time and one can then define a Eulerian pseudomomentum of the wave field that is conserved with density $\tilde{\mathbf{p}}$ and has vertical flux equal to the mean momentum flux $\overline{uw}$. Specifically, one then finds that the first equation in (3) is replaced by (see, e.g., Section 7.1.1 in [5])

$$\tilde{\mathbf{p}}_t + (\overline{uw})_z = 0 \quad \text{with}\ \tilde{\mathbf{p}} = \frac{1}{N^2}\overline{b(u_z - w_x)} - \frac{\overline{b^2}}{2N^4}U_{zz}. \tag{4}$$

For plane waves the generic expression $\tilde{\mathbf{p}} = kE/\hat{\omega}$ holds in terms of the zonal wavenumber k and the intrinsic frequency $\hat{\omega} = \omega - Uk$ and for a plane lee wave with absolute frequency $\omega = 0$ this means $\tilde{\mathbf{p}} = -E/U$. The meaning of $\tilde{\mathbf{p}}$ can be made clearer by using the linear vertical particle displacement $\zeta = -b/N^2$, so that

$$\tilde{\mathbf{p}} = -\overline{\zeta(u_z - w_x)} - \frac{\overline{\zeta^2}}{2}U_{zz}. \tag{5}$$

This makes obvious that the essential meaning of $\tilde{\mathbf{p}}(z, t)$ is an integral of the vorticity over the area between the fixed line of constant z and the material line at $z + \zeta$ whose undisturbed rest position is at z. Kelvin's circulation theorem or a comparison of Reynold's stresses then makes clear that the time-evolving Eulerian zonal mean flow satisfies $\overline{u}_t = \tilde{\mathbf{p}}_t$ and is therefore given by

$$\overline{u}(z, t) = U(z) + \tilde{\mathbf{p}}(z, t) \tag{6}$$

to second order in wave amplitude. This answers the question of where the mean flow energy can change: the kinetic energy density of the zonal mean flow is $\overline{u}^2/2$ and this density changes by the second-order amount $U\tilde{\mathbf{p}}$ when the waves arrive. For a plane lee wave $U\tilde{\mathbf{p}} = -E$ so this is a net loss of mean-flow kinetic energy, which is precisely the amount of energy needed to support the growing wave energy of a developing wave train, as shown in [4]. A sheared region in which $U(z)$ changes does not affect this picture in any essential way, i.e., nothing particularly interesting happens in the shear zone itself.

The existence of a conserved zonal pseudomomentum in the present example is linked to the continuous symmetry of the basic flow with respect to translations in the zonal direction, i.e., the basic flow does not depend on x and this induces a conservation law for the disturbance based on this symmetry (e.g., see [6], for a clear discussion of this link). This is analogous to, but not identical with, the conservation of total fluid momentum induced by the symmetry of the entire fluid system set-up with respect to shifts in the x-direction. Likewise, the conservation of total fluid energy induced by the time symmetry of the total fluid set-up has a counterpart in the conservation of a disturbance-associated *pseudoenergy* that is induced by the time symmetry of the basic flow. In the present case the pseudoenergy density and its vertical flux to second order in wave amplitude are

$$\mathbf{e} = E + U\tilde{\mathbf{p}} \quad \text{and} \quad \overline{pw} + U\,\overline{uw}. \tag{7}$$

This shows that EP's second equation in (3) means that the flux of pseudoenergy is zero. Indeed, the pseudoenergy density itself is zero in the lee wave problem, as is obvious in the plane wave case where (7) reduces to

$$e = E\left(1 + \frac{kU}{\hat{\omega}}\right) = \frac{\omega}{\hat{\omega}}E = 0. \tag{8}$$

So pseudoenergy is conserved but also identically zero in the lee wave problem, which is a reminder that energy budgets in problems with nonzero mean flows are tricky and depend strongly on the chosen reference frame. Indeed, if U is constant then we are at liberty to view the same lee wave problem in a frame of reference moving with the basic flow U. In this reference frame the basic fluid velocity is zero, but the topography is now moving horizontally with velocity $-U$. The pseudoenergy, which is manifestly frame-dependent, is now equal to the wave energy E and the aforementioned extraction of kinetic energy from the basic flow does not take place at second order in wave amplitude. Instead, the moving mountain is now seen as providing the energy for the waves!

In contrast, the pseudomomentum is not frame-dependent and leads to the same mean-flow changes as before. In other words, whilst energy budgets are tricky and frame-dependent, the momentum budget is straightforward and does not depend on the reference frame. This is another lesson that was learned from [1].

3. The trouble with rotation

A student who knows elementary fluid dynamics and encounters rotating stratified flows for the first time might be forgiven for thinking that of the two new features rotation should be trivial to understand compared to stratification. After all, rotating frames are already discussed in undergraduate mechanics books, so this is a seemingly harmless addition. Of course, we soon learn that this is not true at all, and that strong rotation changes the fluid behaviour in a radical way and leads to new effects that are arguably as least as surprising as those to do with stratification.

Rotation certainly makes a big difference for the EP flux in (2), and also for the corresponding definition of pseudomomentum, and for the way in which the mean flow responds to the waves. Indeed, for weak basic flows in the sense of Rossby and Richardson numbers, the EP flux $\boldsymbol{F}$ differs from the familiar Reynolds stresses only by the peculiar term associated with f in the vertical component of $\boldsymbol{F}$. So, why should the familiar vertical Reynold stress be augmented by this peculiar term?

A physical answer to this question was provided by [7], based on earlier work by [8]. Bretherton used the vertical particle displacement $\zeta = -b/N^2$ and the linear zonal momentum equation in a two-dimensional vertical slice model to show that for steady waves

$$-\overline{uw} + \frac{f}{N^2}\,\overline{bv} = -\overline{uw} - f\,\overline{\zeta v} = \overline{\zeta_x p} \qquad (9)$$

holds, where p is the linear perturbation pressure. In other words, the vertical component of the EP flux measures (minus) the *Lagrangian* flux of horizontal momentum across an undulating material surface, which is given by the correlation between the perturbation pressure and the local surface slope ζ_x. This beautiful result also makes clear that it is the EP flux, and not the bare Reynolds stress, that correctly measures the wave drag force on a mountain in a rotating frame. This is obvious because $h = \zeta$ at $z = 0$ in linear theory, where h is the height of the mountain, and therefore the right-hand side of (9) at $z = 0$ is precisely the mean drag on the mountain.

The resolution to the paradox as to how the *Eulerian* momentum flux $\overline{uw}$ and its *Lagrangian* counterpart $-\overline{\zeta_x p}$ can systematically be different hinges on the zonal Coriolis forces exerted on the fluid contained in the pockets between the undulating material surface under consideration and its flat rest position at constant z. Specifically, if the vertical displacement ζ is correlated with the meridional velocity disturbance v then there is a nonzero net zonal Coriolis force at second order in wave amplitude, which precisely compensates for the mismatch between the Eulerian and Lagrangian momentum fluxes. This is basically the physical explanation of the peculiar term associated with f in the EP flux, which captures this correlation.

Now, to find the definition of a rotating pseudomomentum at the level of generality of the situation studied in EP's paper took longer to be worked out [9]. These authors used linear particle displacements in all directions $\boldsymbol{\xi} = (\xi, \eta, \zeta)$, which were defined in terms of the mean material derivative $D_t = \partial_t + U\partial_x$ via

$$D_t\xi = u + \eta U_y + \zeta U_z, \qquad D_t\eta = v, \quad \text{and} \quad D_t\zeta = w. \qquad (10)$$

Note the crucial special form for the zonal displacement ξ, which captures the rate of change of ξ as induced by the *Lagrangian* disturbance velocity, which has to be evaluated to leading order at the displaced position $\boldsymbol{x} + \boldsymbol{\xi}$. With the aid of $\boldsymbol{\xi}$ a suitable three-dimensional rotating generalization of $\tilde{\mathbf{p}}$ could be formulated, but I will not copy it here as it is a lengthy expression consisting of 12 different terms!

This complexity together with the fact that the EP flux had found its most immediate physical explanation in terms of Lagrangian momentum fluxes was also a motivation for developing a general theory of Lagrangian averaging [10,11]. Part of the attraction of that theory lies in the fact that in this theory both a pseudomomentum vector and its associated flux tensor can be defined in a general and fully nonlinear manner that greatly simplifies and also generalizes the small-amplitude Eulerian definitions that were known up to this point. In this way the EP flux could reliably be identified with the Lagrangian-mean flux of zonal mean momentum. Indeed, this was a case where the general finite-amplitude theory was somewhat easier than the small-amplitude theories that had been worked out previously! On the other hand, the simplicity of the theory came at the cost of involving more complicated quantities such as the particle displacements and the Lagrangian-mean velocity, which are not readily available in observations or even in numerical simulations.

So, in practice a combination of the EP flux together with the use of the transformed Eulerian mean equations has been used in atmospheric science ever since it was developed in the late 1970s. This set of equations does not require the explicit knowledge of particle displacements, which is a great practical advantage, and it also makes obvious that it is the divergence of the EP flux that is the essential driver of the zonal mean flow as well as of the concomitant mean meridional circulation in the yz-plane.

Outside Lagrangian-mean theory, the clearest expression of the link between $\nabla\cdot\boldsymbol{F}$ and the forcing of the zonal mean flow can be found in quasi-geostrophic theory, where one can show that the formula

$$\nabla\cdot\boldsymbol{F} = \overline{vq} \quad \text{where } q = v_x - u_y + \frac{f}{N^2}b_z \qquad (11)$$

is the quasi-geostrophic linear potential vorticity disturbance and $\overline{vq}$ is therefore the meridional eddy flux of potential vorticity. The impact of this meridional flux on the mean potential vorticity evolution is quantified by its divergence $(\overline{vq})_y$. Now, this flux divergence is equivalent to the impact of an effective zonal mean force equal to $\overline{vq}$, because the divergence of the meridional eddy flux is then trivially equivalent to the vertical curl of this effective zonal mean force. This makes obvious that $\nabla\cdot\boldsymbol{F}$ can be viewed as an effective zonal mean force exerted on the zonal mean flow, at least in quasi-geostrophic theory. Such a beautiful and clear statement is hard to come by in other situations!

4. Pushing ahead, and into the sea

The results of the EP paper inspired a diverse body of research, which also went beyond the original outline of their theory. For example, a very different methodology for finding conserved wave activity measures and their flux based on Hamiltonian fluid mechanics and the diligent exploitation of basic-flow symmetries was developed more than 25 years after their paper was published. This method used only Eulerian flow variables and yet it generalized easily to finite-amplitude disturbances, which was something that previously only Lagrangian theories had achieved. Applications ran from finite-amplitude wave diagnostics to applications in nonlinear stability (e.g., [12–16]). These finite-amplitude Hamiltonian theories focus on the dynamics of the disturbances and do not necessarily deliver the same amount of information about mean-flow non-acceleration conditions or other aspects of the mean-flow dynamics. However, recently a new direction for finite-amplitude wave activity measures based closely on extensions of the vorticity arguments linking (5) and (6) has been pursued in [17,18], which allows a closer look at both the waves and the mean flow.

Finally, the EP flux and the associated ideas have also found their way into oceanography, which is not as obvious as it sounds, because much of the theory relies on zonal averaging around latitude circles, which fails in most places in the ocean because the continents get in the way! Still, the transformed Eulerian mean formulation and other techniques are now at home on both sides of the air–sea interface [19]. There are also new theoretical ideas that have come out of oceanography, for example [20] greatly extends the notion of thickness-averaged fluid dynamics, which in the atmospheric context had been pursued in the context of EP fluxes and averaging along constant entropy surfaces (e.g., [21]). In summary, the ideas of EP's paper are alive and well, and this surely is a reason to celebrate!

References

[1] A. Eliassen, E. Palm, On the transfer of energy in stationary mountain waves, Geofys Publ. 22 (3) (1960) 1–23.
[2] H. Edmon Jr., B. Hoskins, M. McIntyre, Eliassen–Palm cross sections for the troposphere, J. Atmospheric Sci. 37 (12) (1980) 2600–2616.
[3] J. Charney, P. Drazin, Propagation of planetary-scale disturbances from the lower into the upper atmosphere, J. Geophys. Res. 66 (1) (1961) 83–109.
[4] M.E. McIntyre, M.A. Weissman, On radiating instabilities and resonant overreflection, J. Atmospheric Sci. 35 (1978) 1190–1198.
[5] O. Bühler, Waves and Mean Flows, second ed., Cambridge University Press, 2014. First edition 2009.
[6] W.D. Hayes, Conservation of action and modal wave action, Proc. R. Soc. Lond. A 320 (1970) 187–208.
[7] F.P. Bretherton, Momentum transport by gravity waves, Q. J. R. Meteorol. Soc. 95 (404) (1969) 213–243.
[8] W.L. Jones, Propagation of internal gravity waves in fluids with shear flow and rotation, J. Fluid Mech. 30 (1967) 439–448.
[9] D.G. Andrews, M.E. McIntyre, Planetary waves in horizontal and vertical shear: the generalized Eliassen–Palm relation and the mean zonal acceleration, J. Atmospheric Sci. 33 (1976) 2031–2048.
[10] D.G. Andrews, M.E. McIntyre, An exact theory of nonlinear waves on a Lagrangian-mean flow, J. Fluid Mech. 89 (1978) 609–646.
[11] D.G. Andrews, M.E. McIntyre, On wave-action and its relatives, J. Fluid Mech. 89 (1978) 647–664.
[12] H.D. Abarbanel, D.D. Holm, J.E. Marsden, T.S. Ratiu, Nonlinear stability analysis of stratified fluid equilibria, Philos. Trans. R. Soc. Lond. Ser. A, Math. Phys. Sci. 318 (1543) (1986) 349–409.
[13] M. McIntyre, T. Shepherd, Exact local conservation theorem for finite-amplitude disturbances to non-parallel shear flows, with remarks on Hamiltonian structure and on arnol'd's stability theorems, J. Fluid Mech. 181 (1987) 527–565.
[14] T.G. Shepherd, Symmetries, conservation laws, and Hamiltonian structure in geophysical fluid dynamics, Adv. Geophys. 32 (1990) 287–338.
[15] J. Scinocca, T. Shepherd, Nonlinear wave-activity conservation laws and Hamiltonian structure for the two-dimensional anelastic equations, J. Atmos. Sci. 49 (1) (1992) 5–28.
[16] T. Shaw, T. Shepherd, Wave-activity conservation laws for the three-dimensional anelastic and Boussinesq equations with a horizontally homogeneous background flow, J. Fluid Mech. 594 (2008) 493–506.
[17] N. Nakamura, A. Solomon, Finite-amplitude wave activity and mean flow adjustments in the atmospheric general circulation, part I: quasigeostrophic theory and analysis, J. Atmospheric Sci. 67 (12) (2010) 3967–3983.
[18] N. Nakamura, A. Solomon, Finite-amplitude wave activity and mean flow adjustments in the atmospheric general circulation, part II: analysis in the isentropic coordinate, J. Atmospheric Sci. 68 (11) (2011) 2783–2799.
[19] R.A. Plumb, R. Ferrari, Transformed Eulerian-mean theory, part I: nonquasi-geostrophic theory for eddies on a zonal-mean flow, J. Phys. Oceanogr. 35 (2) (2005) 165–174.
[20] W.R. Young, An exact thickness-weighted average formulation of the Boussinesq equations, J. Phys. Oceanogr. 42 (5) (2012) 692–707.
[21] D.G. Andrews, A finite-amplitude Eliassen–Palm theorem in isentropic coordinates, J. Atmospheric Sci. 40 (8) (1983) 1877–1883.