Constraints on the Mean Mass Transport across Potential Vorticity Contours

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ABSTRACT

An expression is derived for the quasi-horizontal part of the mass transport across a given potential vorticity contour on an isentropic surface, in terms of the rate of change of absolute circulation around the contour and frictional and diabatic terms on the contour. It is deduced that this mass transport is small if the circulation around the contour of interest is steady and if frictional forces and diabatic effects can be neglected on the contour. In a single-layer model the corresponding result is that the total mass transport is zero. In a three-dimensional model the implication is that the dominant mass transport across a vortex edge that tilts in the vertical occurs through vertical advection. It is argued that these constraints on the mass transport are relevant to the estimation of transport across the edge of the stratospheric polar vortex, and the relationship to other similar results that have appeared recently in the literature is discussed. In addition, a new expression is derived for the total mass flux across a three-dimensional surface whose intersection with each isentropic surface is a potential vorticity contour. This expression generalizes previous results that were confined to steady flows and hydrostatic scaling.

1. Introduction

The polar vortex plays an important role as a partial barrier to transport in the lower stratosphere. There has been much interest recently in quantifying the transport across the edge of the vortex, for example, to assess the impact of chemical processing and ozone depletion at high latitudes on the ozone distribution in midlatitudes (e.g., McIntyre 1995; Sobel et al. 1997; Mo et al. 1998).

Since the vortex edge can, under many circumstances, be defined on each isentropic surface by a set of potential vorticity (hereafter PV) contours, there is some advantage in quantifying transport relative to PV contours. What is of most interest for transport is the flux of chemical tracers across the PV contour (or more generally the flux across a surface made up of PV contours on each isentropic surface). Knowing the mass flux across a PV contour is not sufficient to determine the tracer flux across that contour, since there may be variations in the tracer mixing ratio along the contour. Nonetheless, if the mass flux can be estimated, this puts a useful constraint on the tracer flux. A corresponding statement holds for mass and tracer fluxes across a surface made up of PV contours. In calculating the total mass flux across the vortex edge (or any other surface made up of PV contours on each isentropic surface) it is natural to divide the flux, purely geometrically, into two parts. The first part, F_H say, is a quasi-horizontal flux across PV contours that arises from advection along isentropic surfaces. The second part, F_θ say, is a diabatic quasi-vertical flux across isentropes and is generally nonzero for a vortex edge that slopes in the vertical. The total mass flux is then given by

$$F = F_H + F_\theta. \tag{1}$$

The quasi-vertical part F_θ is zero in two-dimensional models, but in three-dimensional models, unless the vortex edge does not slope in the vertical, its contribution to the mass flux can be significant [see Mo et al. (1998) who studied quantitative examples in some idealized theoretical models of steady flows].

In the following we first derive an expression for F_H and show that, under certain plausible assumptions, it is small, indeed much smaller than a naive estimate based on typical horizontal velocities would suggest. The results we discuss are simply based on the balance of mass and potential vorticity within appropriate control volumes and do not require new coordinates to be defined. However, we note that Nakamura (1995) has set out in some detail a formalism for quantifying trans-

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port relative to chemical tracer contours or PV contours by making an explicit coordinate transformation (to an ''equivalent latitude'' coordinate). We then derive general expressions for both parts of F through a three-dimensional vortex edge. These general expressions do not rely on isentropic coordinates or on hydrostatic scaling, and they allow a useful cross-check on our first result and on results published elsewhere.

2. The mass transport relation for F_H

We use the hydrostatic primitive equations in isentropic coordinates and the usual hydrostatic expression $Q \equiv (\partial v_a/\partial x - \partial u_a/\partial y)/\sigma$ for the potential vorticity, where $\mathbf{u}_a = (u_a, v_a, 0)$ is the absolute horizontal velocity (i.e., including velocity contributions from the earth's rotation), σ is the isentropic density, and where the derivatives are performed at constant potential temperature θ. On each isentropic surface, labeled by θ, we choose a particular PV contour $\Gamma(\theta, t)$ that lies in the vortex edge on that surface. We assume that on $\Gamma(\theta, t)$ the PV, Q, is equal to $Q_\Gamma(\theta)$. The absolute circulation $\mathcal{C}(\theta, t)$ around $\Gamma(\theta, t)$ satisfies the identity

$$\mathcal{C}(\theta, t) \equiv \oint_{\Gamma(\theta,t)} \mathbf{u}_a \cdot \mathbf{ds} = \iint_{\mathcal{A}_\Gamma} \sigma Q \, dx \, dy. \tag{2}$$

The second integral in (2) is taken over the area $\mathcal{A}_\Gamma$ enclosed by $\Gamma(\theta, t)$.

Using the second equality in (2), the time derivative of $\mathcal{C}(\theta, t)$ can then be evaluated as

$$\frac{\partial \mathcal{C}(\theta, t)}{\partial t} = \iint_{\mathcal{A}_\Gamma} \frac{\partial(\sigma Q)}{\partial t} \, dx \, dy + \oint_{\Gamma(\theta,t)} \sigma Q \mathbf{u}_\Gamma \cdot \hat{\mathbf{n}} \, ds, \tag{3}$$

where $\mathbf{u}_\Gamma$ is the horizontal velocity of the PV contour Γ normal to itself and where $\hat{\mathbf{n}}$ is the horizontal outward unit vector normal to the PV contour. The first term on the right-hand side can be rewritten using the flux form of the PV evolution equation,

$$\frac{\partial(\sigma Q)}{\partial t} + \boldsymbol{\nabla} \cdot [\sigma Q \mathbf{u}_H] + \boldsymbol{\nabla} \cdot \mathbf{J} = 0, \tag{4}$$

where $\mathbf{u}_H = (u, v, 0)$ is the horizontal velocity and

$$\mathbf{J} \equiv \underbrace{\left(\dot{\theta}\frac{\partial v}{\partial \theta}, -\dot{\theta}\frac{\partial u}{\partial \theta}, 0\right)}_{\equiv \mathbf{J}_{\dot{\theta}}} + \underbrace{(-G_2, G_1, 0)}_{\equiv \mathbf{J}_\mathbf{G}} \tag{5}$$

is the nonadvective PV flux defined by Haynes and McIntyre (1987, hereafter HM87). The diabatic part $\mathbf{J}_{\dot{\theta}}$ depends on diabatic heating $\dot{\theta} \equiv D\theta/Dt$ as well as on vertical shear. The part $\mathbf{J}_\mathbf{G}$ due to a nonconservative body force, $\mathbf{G}$, is simply given in terms of the horizontal components G_1 and G_2 of that force. Note that $\mathbf{J}$ has no cross-isentropic component. In real three-dimensional flow the part $\mathbf{J}_\mathbf{G}$ is perhaps due to gravity wave dissipation or to vertical momentum transport by localized patches of turbulence or to the direct effects of molecular diffusion of momentum. The other part, $\mathbf{J}_{\dot{\theta}}$, is associated with quasi-vertical advection of momentum by the diabatic, cross-isentropic velocity.

Substituting in (3) and using the two-dimensional divergence theorem then gives

$$\frac{\partial \mathcal{C}(\theta, t)}{\partial t} = \oint_{\Gamma(\theta,t)} \sigma Q (\mathbf{u}_\Gamma - \mathbf{u}_H) \cdot \hat{\mathbf{n}} \, ds - \oint_{\Gamma(\theta,t)} \mathbf{J} \cdot \hat{\mathbf{n}} \, ds \tag{6}$$

$$= Q_\Gamma(\theta) \oint_{\Gamma(\theta,t)} \sigma (\mathbf{u}_\Gamma - \mathbf{u}_H) \cdot \hat{\mathbf{n}} \, ds - \oint_{\Gamma(\theta,t)} \mathbf{J} \cdot \hat{\mathbf{n}} \, ds, \tag{7}$$

where $Q_\Gamma(\theta)$ can be moved outside the integral because $Q = Q_\Gamma(\theta)$ everywhere along the PV contour. The first integral on the right-hand side represents the PV accumulation inside the contour due to the advection of PV across the contour with relative velocity $(\mathbf{u}_\Gamma - \mathbf{u}_H)$. Because of the fact that Q is constant and equal to $Q_\Gamma(\theta)$ on the contour, this part of the rate of change is precisely equal to $Q_\Gamma(\theta)$ multiplied by the horizontal mass flux F_H (per unit ''vertical'' depth as measured by a unit step in potential temperature) across the contour, defined by

$$F_H(\theta, t) \equiv \oint_{\Gamma(\theta,t)} \sigma (\mathbf{u}_\Gamma - \mathbf{u}_H) \cdot \hat{\mathbf{n}} \, ds. \tag{8}$$

Here, F_H is defined to be positive for flux *into* the vortex.

We see that combining (8) with (6) yields

$$\frac{\partial \mathcal{C}(\theta, t)}{\partial t} = Q_\Gamma(\theta) F_H - \oint_{\Gamma(\theta,t)} \mathbf{J} \cdot \hat{\mathbf{n}} \, ds, \tag{9}$$

relating the quasi-horizontal mass flux across the PV contour and changes of the circulation around the PV contour. [This result has been noted independently by Thuburn and Lagneau (1999).]

Note that all that we have written so far applies with minor modifications to a single-layer system, for example, to a suitable shallow water numerical model. In such a single-layer system the nonadvective PV flux due to forcing $\mathbf{J}_\mathbf{G}$ is the same as in (5), that is, it is equal to $\mathbf{k} \times \mathbf{G}$, where $\mathbf{k}$ is the unit vector perpendicular to the layer and $\mathbf{G}$ is any nonconservative body force, for example, friction, perhaps associated with a model small-scale hyperdiffusion. The diabatic effects are usually modeled simply by adding a suitable mass source–

sink term in the single-layer continuity equation, while neglecting the corresponding momentum source–sink terms in the momentum equations. (This corresponds to the injection or extraction of fluid with velocity exactly equal to the local fluid velocity.) In other words, the diabatic PV flux $\mathbf{J}_{\dot\theta}$ in (5), which is due to the quasi-vertical transport of momentum, is usually neglected in a single-layer system.

In many numerical simulations of the stratospheric polar vortex, both two-dimensional and three-dimensional simulations (e.g., Juckes 1989; Haynes 1990), it appears that the flow settles down to a quasi-equilibrium state in which the vortex edge is defined over a substantial length of time by a set of PV contours. If we take one such PV contour, on a given isentropic surface, we might reasonably argue that the circulation around it is steady and hence deduce that in (9) the left-hand side is zero and hence that the mass flux across the contour is proportional to the nonadvective flux contribution, that is

$$F_H(\theta, t) = \frac{1}{Q_\Gamma(\theta)} \oint_{\Gamma(\theta,t)} \mathbf{J} \cdot \hat{\mathbf{n}}\, ds. \qquad (10)$$

Even if the circulation were not exactly steady, a similar equation could be derived in the time average. We note the following results that follow from (10).

1) If $\mathbf{J}$ were identically zero on Γ, then (10) would imply that the net quasi-horizontal mass flux F_H (or its time average) across Γ is identically zero.
2) In three-dimensional flow, if the body force $\mathbf{G}$ vanishes on the contour, then the net quasi-horizontal mass flux across Γ is proportional to $\oint \mathbf{J}_{\dot\theta} \cdot \hat{\mathbf{n}}\, ds$. As noted by HM87, under quasi-geostrophic scaling the flux $\mathbf{J}_{\dot\theta}$ is small in comparison with typical isentropic advective fluxes of PV, essentially because of the smallness of the vertical velocity compared to naive estimates from the mass-continuity equation. Thus the net quasi-horizontal mass flux across Γ is smaller by a factor of Ro than a naive order-of-magnitude estimate, also based on quasi-geostrophic theory, that is simply based on typical isentropic velocities and, hence, does not recognize (10) explicitly.
3) More generally, using the estimate $|\mathbf{J}_{\dot\theta}| \sim \mathrm{Ro}U^2/L \sim \mathrm{Ro}^2 Uf$ given by HM87, where Ro is the Rossby number, U is a typical velocity, and L is a typical horizontal length scale, we have

$$F_H \sim \sigma L \max\left(\mathrm{Ro}^2 U, \frac{\mathcal{G}}{f}\right), \qquad (11)$$

where $\mathcal{G}$ is a typical magnitude for $\mathbf{G}$ and f is a typical magnitude for the Coriolis parameter.

3. Mass transport relations for both parts of *F* on three-dimensional vortex edges

We now consider the total mass flux into the vortex F, which by (1) is the sum of the previously considered quasi-horizontal flux F_H and the quasi-vertical flux F_θ. An explicit expression for F_θ valid for finite-amplitude undulations of the vortex edge was given by Mo et al. (1998), which in Cartesian coordinates is [cf. their (4.3)]

$$F_\theta = \iint \rho \left(\frac{\partial\theta}{\partial z}\right)^{-1} \dot\theta\, dx\, dy. \qquad (12)$$

Here, the integral is performed over the edge surface of the vortex, and $dx\, dy$ denotes the horizontal projection of the surface element, defined to be positive if the edge surface slopes outward with increasing altitude. Using standard quasigeostrophic scaling arguments, it can be shown that the flux (12) across a sloping vortex edge is $O(\mathrm{Ro}^{-1})$ larger than the horizontal flux (11), if $\mathcal{G}$ is zero. However, the analysis leading to (12) in Mo et al. (1998) was restricted to steady flows and hydrostatic scaling.

It is possible to derive a (coordinate-independent) expression for both parts of the mass flux $F = F_H + F_\theta$ without these restrictions, as we shall show now. This makes it self-evident that there are no essential complications associated with allowing unsteady flows and nonhydrostatic scaling.

We now use the three-dimensional Rossby–Ertel PV defined by $Q \equiv (\boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}\theta)/\rho$, where the absolute vorticity $\boldsymbol{\zeta}_a$ and the gradient operator $\boldsymbol{\nabla}$ now have the standard meaning in three-dimensional flow and where ρ is density in Cartesian coordinates. As before, it is convenient to use the flux form of the PV equation, now written in standard Cartesian coordinates as [e.g., HM87, Eq. (3)]

$$\frac{\partial(\rho Q)}{\partial t} + \boldsymbol{\nabla} \cdot [\rho Q \mathbf{u}] = \boldsymbol{\nabla} \cdot [\dot\theta \boldsymbol{\zeta}_a + \mathbf{G} \times \boldsymbol{\nabla}\theta]. \qquad (13)$$

Multiplying (13) by $Q_\Gamma(\theta)^{-1}$, it may be shown, using the definition of $\dot\theta$ as the material rate of change of θ, that

$$\frac{\partial}{\partial t}\left(\frac{\rho Q}{Q_\Gamma(\theta)}\right) + \boldsymbol{\nabla} \cdot \left[\frac{\rho Q}{Q_\Gamma(\theta)}\mathbf{u}\right] = \boldsymbol{\nabla} \cdot \left[\frac{\dot\theta \boldsymbol{\zeta}_a}{Q_\Gamma(\theta)} + \frac{\mathbf{G} \times \boldsymbol{\nabla}\theta}{Q_\Gamma(\theta)}\right]. \qquad (14)$$

Note that this is simply a restatement of the PV equation (13) except that the freedom allowed in the definition of PV (HM87) has been exploited and the PV has been redefined by multiplying the original by $Q_\Gamma(\theta)^{-1}$.

We now integrate this equation over the volume V defined by $Q \geq Q_{\Gamma(\theta)}$, $\theta_B \leq \theta \leq \theta_T$ (see Fig. 1). The bounding surface $\partial\mathcal{V}$ of this volume is made up of the surface $Q = Q_{\Gamma(\theta)}$, which we might denote by $\partial\mathcal{V}_Q$ and the remainder, say, $\partial\mathcal{V}_\theta$, comprising the isentropic surface $\theta = \theta_T$ and the isentropic surface $\theta = \theta_B$, the latter only if the contour $\Gamma(\theta, t)$ still exists on this isentropic surface (i.e., it lies above the bottom of the vortex).

Integrating (14) over this region and then taking the time derivative outside the integral gives

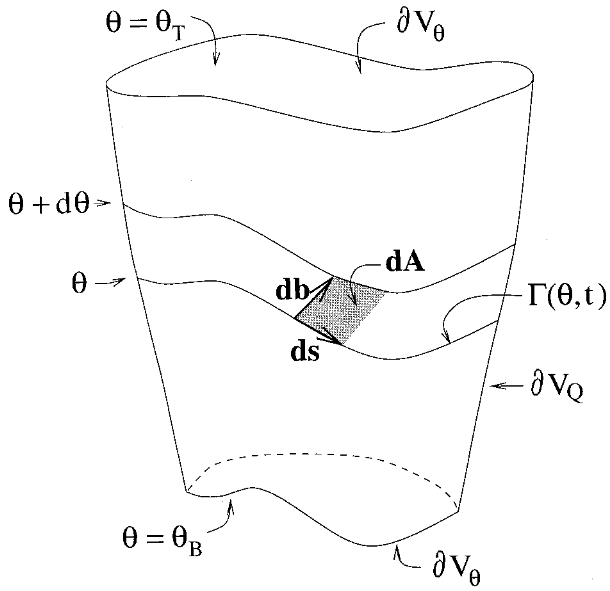

FIG. 1. Schematic snapshot of a three-dimensional vortex. The top and bottom boundaries (denoted by $\partial\mathcal{V}_\theta$) are given by the isentropes $\theta = \theta_T$ and $\theta = \theta_B$, respectively. The horizontal boundary $\partial\mathcal{V}_Q$ is marked by the isentropic contours $\Gamma(\theta, t)$. Also indicated is the surface band between two isentropes θ and $\theta + d\theta$, the corresponding vector increments **ds** and **db**, and the area element $\mathbf{dA} = \mathbf{ds} \times \mathbf{db}$, all of which are used in the appendix. Note that the (infinitesimal) width of the band is strongly exaggerated for illustration.

$$\frac{d}{dt}\int_{\mathcal{V}} \frac{\rho Q}{Q_\Gamma(\theta)}\,dV + \int_{\partial\mathcal{V}} \frac{\rho Q}{Q_\Gamma(\theta)}(\mathbf{u} - \mathbf{u}_e)\cdot\mathbf{n}\,dA = \int_{\partial\mathcal{V}} \frac{\dot\theta\boldsymbol{\zeta}_a}{Q_\Gamma(\theta)}\cdot\mathbf{n}\,dA + \int_{\partial\mathcal{V}} \frac{\mathbf{G}\times\nabla\theta}{Q_\Gamma(\theta)}\cdot\mathbf{n}\,dA, \quad (15)$$

where $\mathbf{n}$ is the outward-pointing unit normal vector and $\mathbf{u}_e$ is the velocity of the bounding surface normal to itself.

Now consider the contributions from the various surface integrals over the vertical part $\partial\mathcal{V}_\theta$ of the bounding surface, on which $\mathbf{n}$ is parallel to $\boldsymbol{\nabla}\theta$. The contribution from the second term on the right-hand side disappears because two of the vectors in the triple product are parallel. Since $\partial\mathcal{V}_\theta$ is made up of isentropic surfaces, it must be that, on $\partial\mathcal{V}_\theta$, $\theta_t + \mathbf{u}_e\cdot\boldsymbol{\nabla}\theta = 0$ and, hence, that $(\mathbf{u} - \mathbf{u}_e)\cdot\boldsymbol{\nabla}\theta = \dot\theta$. Using the identity $\rho Q = \boldsymbol{\zeta}_a\cdot\boldsymbol{\nabla}\theta$ it may be shown that the contributions from the second integral on the left-hand side and from the first integral on the right-hand side cancel on $\partial\mathcal{V}_\theta$. It follows that the surface integrals in (15) need only be evaluated over $\partial\mathcal{V}_Q$, that is, on those parts for which $Q = Q_\Gamma(\theta)$.

Hence (15) reduces to

$$\frac{d}{dt}\int_{\mathcal{V}} \frac{\rho Q}{Q_\Gamma(\theta)}\,dV + \int_{\partial\mathcal{V}_Q} \rho(\mathbf{u} - \mathbf{u}_e)\cdot\mathbf{n}\,dA = \int_{\partial\mathcal{V}_Q} \frac{\dot\theta\boldsymbol{\zeta}_a}{Q_\Gamma(\theta)}\cdot\mathbf{n}\,dA + \int_{\partial\mathcal{V}_Q} \frac{\mathbf{G}\times\boldsymbol{\nabla}\theta}{Q_\Gamma(\theta)}\cdot\mathbf{n}\,dA. \quad (16)$$

It is shown in the appendix how the volume integral can be written as a surface integral involving the isentropic circulations $\mathcal{C}(\theta, t)$. If we assume that $\mathcal{C}$ on each isentrope is constant in time, then we may equate the total mass flux to the terms on the right-hand side. In doing this it is useful to first write $\mathbf{n}dA = \mathbf{dA}_H + \mathbf{dA}_\theta$, where $\mathbf{dA}_H$ satisfies $\boldsymbol{\nabla}\theta\cdot\mathbf{dA}_H = 0$ and $\mathbf{dA}_\theta$ satisfies $\boldsymbol{\nabla}\theta\times\mathbf{dA}_\theta = 0$. Then we have, after a sign change,

$$\int_{\partial\mathcal{V}_Q} \rho(\mathbf{u}_e - \mathbf{u})\cdot\mathbf{n}\,dA = -\int_{\partial\mathcal{V}_Q} \frac{\dot\theta\boldsymbol{\zeta}_a}{Q_\Gamma(\theta)}\cdot\mathbf{dA}_H - \int_{\partial\mathcal{V}_Q} \frac{\mathbf{G}\times\boldsymbol{\nabla}\theta}{Q_\Gamma(\theta)}\cdot\mathbf{dA}_H - \int_{\partial\mathcal{V}_Q} \frac{\dot\theta\boldsymbol{\zeta}_a}{Q_\Gamma(\theta)}\cdot\mathbf{dA}_\theta. \quad (17)$$

This expression is equal to the total mass flux F. The first two terms on the right-hand side are just those considered earlier for F_H, and in particular the first can be argued to be small under small-Ro scaling. The third term is F_θ, and it may be rewritten using the fact that $\mathbf{dA}_\theta$ is parallel to $\boldsymbol{\nabla}\theta$ as

$$F_\theta = -\int_{\partial\mathcal{V}_Q} \mathrm{sgn}(\boldsymbol{\nabla}\theta\cdot\mathbf{dA}_\theta)\frac{\rho\dot\theta}{|\boldsymbol{\nabla}\theta|}\,|\mathbf{dA}_\theta|, \quad (18)$$

where $\mathrm{sgn}(\boldsymbol{\nabla}\theta\cdot\mathbf{dA}_\theta)$ gives a plus or minus sign according to whether $\boldsymbol{\nabla}\theta$ and $\mathbf{dA}_\theta$ point into the same or into opposite directions. This gives a minus sign in the generic case, in which the vortex edge slopes outward with increasing altitude while θ increases with altitude. Finally, replacing $\boldsymbol{\nabla}\theta$ with $\partial\theta/\partial z$ and $-\mathrm{sgn}(\boldsymbol{\nabla}\theta\cdot\mathbf{dA}_\theta)|\mathbf{dA}_\theta|$ with $dx\,dy$, we recover (12), that is, the hydrostatic result of Mo et al. (1998).

4. Discussion

The results in this paper leading to (11) show that under certain circumstances the horizontal mass flux F_H across a PV contour on an isentropic surface is a factor of $O(\mathrm{Ro})$ smaller than might be expected on the basis of naive scaling arguments based simply on typical horizontal velocities. Similar results, usually starting from (4) and assuming at that stage that the flow is in steady state, have been noted by others and are reviewed by Sobel and Plumb (1999). Such results include those of Schneider (1987), McIntyre and Norton (1990), and Mo et al. (1998). Sobel and Plumb's (1999 hereafter SP) result is more general, in that the flow is not assumed to be perfectly steady, but the mass enclosed within each PV contour that lies within Γ is assumed to be steady. Using the additional assumption that $\mathbf{J} = 0$ everywhere, these authors showed that the net mass source within all contours must be zero, which implies that $F_H = 0$ across all contours. Our results, following from (9) above, focus directly on F_H across the bounding contour

Γ and are based on a different set of assumptions, namely on the steadiness of the circulation $\mathcal{C}$ and on the vanishing of $\mathbf{J}$ only on Γ itself. Situations can occur in which both sets of assumptions apply, but we regard our results and those of SP as mainly complementary. The result (9) has been noted independently by Thuburn and Lagneau (1999), but they then derive constraints on the mass source within the contour, rather than the mass flux across the contour.

Unlike F_H, the vertical flux F_θ can only be computed using a three-dimensional vortex edge. This part of the flux has received considerably less attention than F_H, and here our results leading to (18) offer a useful cross-check and generalization of the results of Mo et al. (1998).

In conclusion, we return to the question of whether (10) is useful only when it is applied to contours in the vortex edge. The simplest form of our results applies when the circulation around Γ is exactly steady. We have argued that this is likely to be a good approximation for contours in the vortex edge in a flow that has reached a statistical equilibrium. Of course, (10) should apply in the time average to any contour, including contours in the surf zone, if the flow has reached statistical equilibrium. There might, however, be circumstances under which we can argue that the flux $\mathbf{J}$ is systematically smaller in the vortex edge than in the surf zone. For example, this seems plausible in a single-layer numerical model, where $\mathbf{J}$ is associated with small-scale dissipation, which is likely to be most active in the surf zone. Under such conditions it follows that the time average mass flux across PV contours in the vortex edge will be systematically less than that across PV contours in the surf zone. Explicit single-layer numerical calculations that demonstrate such behavior are presented and discussed in some detail in SP. However, in realistic flows, where $\mathbf{J}$ might be due to gravity wave drag, for example, then it would seem to be far more difficult to argue that the time average of the right-hand side of (10) would be systematically less in the vortex edge than in the surf zone.

A final point concerns the relation of mass transport and tracer transport. The total tracer flux may be expressed as the sum of two terms, the first the product of the average tracer mixing ratio and the the total mass flux, the second an average of the product of local fluctuations in mass flux and tracer mixing ratio around the contour. The results here constrain the size of the first term and suggest that it is smaller on the vortex edge compared to the surf zone. However, it is difficult to make any prediction of the comparative size of the second term in the vortex edge or in the surf zone, except to note that in the vortex edge one might expect good correlations between PV and tracer contours and, hence, that the fluctuations in tracer mixing ratio around PV contours are quite small there.

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APPENDIX

The PV Volume Integral in (16)

Using the identity $\rho Q \equiv (\boldsymbol{\nabla} \times \mathbf{u}_a) \cdot \boldsymbol{\nabla}\theta$, the PV volume integral in (16) can be converted to a surface integral given by (cf. HM87)

$$\int_{\mathcal{V}} \frac{(\boldsymbol{\nabla} \times \mathbf{u}_a) \cdot \boldsymbol{\nabla}\theta}{Q_\Gamma(\theta)}\, dV = \int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{u}_a \times \boldsymbol{\nabla}\theta}{Q_\Gamma(\theta)}\right) dV = \int_{\partial\mathcal{V}_Q} \left(\frac{\mathbf{u}_a \times \boldsymbol{\nabla}\theta}{Q_\Gamma(\theta)}\right) \cdot \mathbf{dA}. \quad \text{(A1)}$$

Here, the total boundary $\partial\mathcal{V}$ has already been replaced by the horizontal boundary $\partial\mathcal{V}_Q$ because the integrand vanishes at the isentropic top and bottom boundaries $\partial\mathcal{V}_\theta$.

The contribution to the surface integral in (A1) from a surface band bounded by two infinitesimally close isentropes, θ and $\theta + d\theta$, is now calculated (cf. Fig. 1). The surface element $\mathbf{dA}$ can be written as $\mathbf{dA} = \mathbf{ds} \times \mathbf{db}$, where $\mathbf{ds}$ and $\mathbf{db}$ are vector increments lying inside the surface band. The vector increment $\mathbf{ds}$ is the line element along the isentropic edge contour $\Gamma(\theta, t)$, and the vector increment $\mathbf{db}$ connects the two isentropes. The relations $\mathbf{ds} \cdot \boldsymbol{\nabla}\theta = 0$ and $d\theta = \mathbf{db} \cdot \boldsymbol{\nabla}\theta$ therefore hold by construction.

Hence, the relevant integrand in (A1) can be written as

$$\left(\frac{\mathbf{u}_a \times \boldsymbol{\nabla}\theta}{Q_\Gamma(\theta)}\right) \cdot \mathbf{dA} = \frac{(\mathbf{u}_a \cdot \mathbf{ds})(\boldsymbol{\nabla}\theta \cdot \mathbf{db})}{Q_\Gamma(\theta)} = \mathbf{u}_a \cdot \mathbf{ds} \frac{d\theta}{Q_\Gamma(\theta)}, \quad \text{(A2)}$$

and the contribution from the infinitesimal surface band is given by a contour integral around $\Gamma(\theta, t)$ with integrand (A2). The contributions arising from many such infinitesimal surface bands can then be integrated in θ to yield

$$\int_{\partial\mathcal{V}_Q} \left(\frac{\mathbf{u}_a \times \boldsymbol{\nabla}\theta}{Q_\Gamma(\theta)}\right) \cdot \mathbf{dA} = \int_{\theta_B}^{\theta_T} \left[\oint_{\Gamma(\theta,t)} \mathbf{u}_a \cdot \mathbf{ds}\right] \frac{d\theta}{Q_\Gamma(\theta)}. \quad \text{(A3)}$$

Substituting (A3) in (A1) then gives the final result

$$\int_{\mathcal{V}} \frac{\rho Q}{Q_\Gamma(\theta)}\, dV = \int_{\theta_B}^{\theta_T} \mathcal{C}(\theta, t)\, \frac{d\theta}{Q_\Gamma(\theta)}, \quad \text{(A4)}$$

where $\mathcal{C}(\theta, t)$ is the absolute circulation around the edge contour $\Gamma(\theta, t)$ in (A3). It is noteworthy that the identity (A4) holds for arbitrary functions of potential temperature $Q_\Gamma(\theta)$ and that it has *not* been necessary to assume here that the edge contours $\Gamma(\theta, t)$ are contours of constant PV. However, such an assumption is needed to relate the time derivative of (A4) to the mass flux, as was demonstrated below (16).

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