Quiz 3 - Solution

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1. Jim plays a game by rolling a pair of fair 6 sided dice and following ALL these rules:
   • If doubles come up, he gets that value in $'s. (e.g. if 6-6 comes up, he gets $6)
   • If both dice are even, he gets $2.
   • If either dice shows a 6, he loses $4.
   What is the expected value of the money he loses/gains in this game? Does he gain or lose money in the long run by playing this game?

1 Solution 1 - Use linearity of expectation

Let $X$ be the random variable which is Joe’s gain/loss in one round. Since there are three rules to be applied, it is natural to decompose $X$ as:

$$X = Y_1 + Y_2 + Y_3$$

where:

$$Y_1 = \text{gain/loss from Rule 1}$$
$$Y_2 = \text{gain/loss from Rule 2}$$
$$Y_3 = \text{gain/loss from Rule 3}$$

We now analyze these three random variables separately, and then put them together afterward to understand $E[X]$.

1.0.1 $Y_1$

$Y_1$ is the contribution from Rule 1, so we have:

$$Y_1 = \begin{cases} 
1 & \text{if (1,1) comes up} \\
2 & \text{if (2,2) comes up} \\
3 & \text{if (3,3) comes up} \\
4 & \text{if (4,4) comes up} \\
5 & \text{if (5,5) comes up} \\
6 & \text{if (6,6) comes up} \\
0 & \text{on all other outcomes}
\end{cases}$$

Thus:

$$E[Y_1] = 0 \cdot P(Y_1 = 0) + 1 \cdot P(Y_1 = 1) + 2 \cdot P(Y_1 = 2) + \ldots + 6 \cdot P(Y_1 = 6)$$

$$= 0 + \frac{1}{36} + \frac{2}{36} + \ldots + \frac{6}{36} = \frac{21}{36}$$

1.0.2 $Y_2$

$Y_2$ is the contribution from Rule 2, so we have:

$$Y_2 = \begin{cases} 
2 & \text{if both dice are even} \\
0 & \text{on all other outcomes}
\end{cases}$$
Define $E$ to be the event $E = \{\text{both dice are even}\}$, so that $Y_2$ is 2 exactly on $E$. (i.e. $E = \{Y_2 = 2\}$.) We know $\Pr (E) = \frac{1}{4}$ since each die has probability 1/2 to be even, and the two dice rolls are independent. (Alternatively, there are $3 \cdot 3 = 9$ ways both die can be even, and 36 total outcomes, so $\Pr (E) = 9/36 = 1/4$. Thus:

$$
\mathbb{E} [Y_2] = 0 \Pr (Y_2 = 0) + 2 \Pr (Y_2 = 2) = 0 + 2 \Pr (E) = 2 \cdot \left(\frac{1}{4}\right) = \frac{18}{36}
$$

(Remark: $Y_2$ can be nicely written using indicator function. We could write $Y_2 = 2 \cdot I_E$ where $I_E$ is the indicator of $E$. Then $\mathbb{E} [Y_2] = \mathbb{E} [2I_E] = 2 \Pr (E)$ is clear by linearity of expectation,...this notation can make writing $Y_2$ nicer but is not required for this problem).

1.0.3 $Y_3$

$Y_3$ is the contribution from Rule 3, so we have:

$$Y_3 = \begin{cases} -4 & \text{if either die is a 6} \\ 0 & \text{on all other outcomes} \end{cases}$$

Define $F$ to be the event $F = \{\text{either die is a 6}\}$, so that $Y_3$ is -4 exactly on $F$ (i.e. $F = \{Y_3 = -4\}$). We know that

$$
\Pr (F) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36},
$$

by the inclusion-exclusion principle since each die has a 1/6 chance to come up 6, and there is a 1/36 chance BOTH dice come up 6. (Another way to do this is to notice that $F^c$ is the event that BOTH dice are NOT 6, and so we have $\Pr (F) = 1 - \Pr (F^c) = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$ since each die has a 5/6 chance to NOT come up 6 and the two dice are independent.) Thus:

$$
\mathbb{E} [Y_3] = 0 \Pr (Y_2 = 0) - 4 \Pr (Y_2 = -4) = 0 - 4 \Pr (F) = -4 \left(\frac{11}{36}\right) = -\frac{44}{36}
$$

(Remark: $Y_3$ can be nicely written using indicator function. We could write $Y_2 = -4 \cdot I_F$ where $I_F$ is the indicator of $F$. Then $\mathbb{E} [Y_2] = \mathbb{E} [-4I_F] = -4 \Pr (F)$ is clear by linearity of expectation)

1.0.4 Putting it together

By linearity of expectation, we have that:

$$
\mathbb{E} [X] = \mathbb{E} [Y_1 + Y_2 + Y_3] = \mathbb{E} [Y_1] + \mathbb{E} [Y_2] + \mathbb{E} [Y_3] = \frac{21}{36} + \frac{18}{36} - \frac{44}{36} = -\frac{5}{36}
$$

Since this is negative Joe loses money in the long run.

1.1 Common Errors

The most fundamental error made following this approach was to confuse $\text{RANDOM VARIABLES and EVENTS}$. For example, $E$ is an event; you cannot take the expected value of an event, i.e. $\mathbb{E} [E]$ makes no sense. The allowed operation is to take the probability of the event, $\Pr (E)$. (Remark: If you like indicator functions, then you are allowed to do $\mathbb{E} [I_E] = \Pr (E)$...this part of the joy of indicator functions!) Another example: if $Y_2$ is a random variable, you cannot take the probability of this, i.e. $\Pr (Y_2)$ makes no sense. The allowed operation is to take the expected value of $Y_2$, e.g. $\mathbb{E} [Y_2]$ or to ask for the probability that $Y_2$ takes a specific value e.g. $\Pr (Y_2 = 2)$. Make sure you understand the difference between random variables and events!

Some people also did a similar calculation to the above without really saying what $Y_1, Y_2, Y_3$ were....in any solution it should be clear why the linearity of expectation justifies the formula for $\mathbb{E} [X]$. (For example it is not true that $X$ takes the value +2 with probability 1/2, so writing $\mathbb{E} [X] = \ldots + 2 \cdot (\frac{1}{2}) + \ldots$ without further justification is misleading, even though this does appear as part of the calculation using $Y_2$ as above)
2 Solution 2 - List ALL the possible outcomes

Let $X$ be the random variable which is Joe’s gain/loss in one round. One way to find the expected value is to explicitly list all the outcomes of the two dice and the value of $X$ at that outcome. There are 36 outcomes, and three different rules to be applied, so this takes a bit of effort. The easiest way to organize this is in a table, where the $i$th row $j$-th column represents the outcomes where the first dice came up $i$ and the second dice came up $j$ (e.g. the top right corner represents the outcomes $(1,1)$). Since there are three rules, each box can be the sum of up to three values (I leave it blank to represent a 0):

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 &  &  & -4 & \\
2 & 2 & 2 &  &  & 2-4 \\
3 &  & 3 &  & -4 & \\
4 &  & 4 & 2 &  & 2-4 \\
5 &  &  & 5 &  & -4 \\
6 & -4 & 2-4 & -4 & 2-4 & 6-24 \\
\end{array}
$$

If you do the arithmetic this is the same as:

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 &  &  & -4 & \\
2 & 4 & 2 &  & -2 & \\
3 &  & 3 &  & -4 & \\
4 & 2 &  & 6 &  & -2 \\
5 &  &  &  & 5 & -4 \\
6 & -4 & -2 & -4 & -2 & 4 \\
\end{array}
$$

So we see that $X$ always takes one of the values in $\{-4, -2, 0, 1, 2, 3, 4, 5, 6\}$ and we can compute each of the probabilities by counting the number of occurrences in the above table. (All outcomes are equally likely, so the probability is equal to the number of occurrences divided by 36). Finally then, we can get the expected value by:

$$
E[X] = (-4) \cdot P(X = -4) + (-2) \cdot P(X = -2) + 0 \cdot P(X = 0) \\
+ 1 \cdot P(X = 1) + 2 \cdot P(X = 2) + 3 \cdot P(X = 3) + 4 \cdot P(X = 4) \\
+ 5 \cdot P(X = 5) + 6 \cdot P(X = 6)
$$

$$
= (-4) \left( \frac{6}{36} \right) + (-2) \left( \frac{4}{36} \right) + 0 \left( \frac{18}{36} \right) \\
+ 1 \left( \frac{1}{36} \right) + 2 \left( \frac{2}{36} \right) + 3 \left( \frac{1}{36} \right) + 4 \left( \frac{2}{36} \right) \\
+ 5 \left( \frac{1}{36} \right) + 6 \left( \frac{1}{36} \right)
$$

$$
= -\frac{32}{36} + \frac{27}{36} = -\frac{5}{36}
$$

Since this is negative, Joe loses money in the long run.

2.1 Common Errors

Many people did not organize their calculation here well and ended up making many errors…if the calculation will be messy as it is here you should try to have a systematic way of organizing it to avoid double counting outcomes.

Also you should be warned that this kind of explicitly writing out is hard to do as the size of the dice get larger. (e.g. for a 20 sided dice it would take you a lot longer to do this and there is even more potential for errors). For this this reason the first solution is a bit nicer and a bit more general.