

Advanced Machine Learning

Learning with Large Expert Spaces

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Problem

- Learning guarantees:

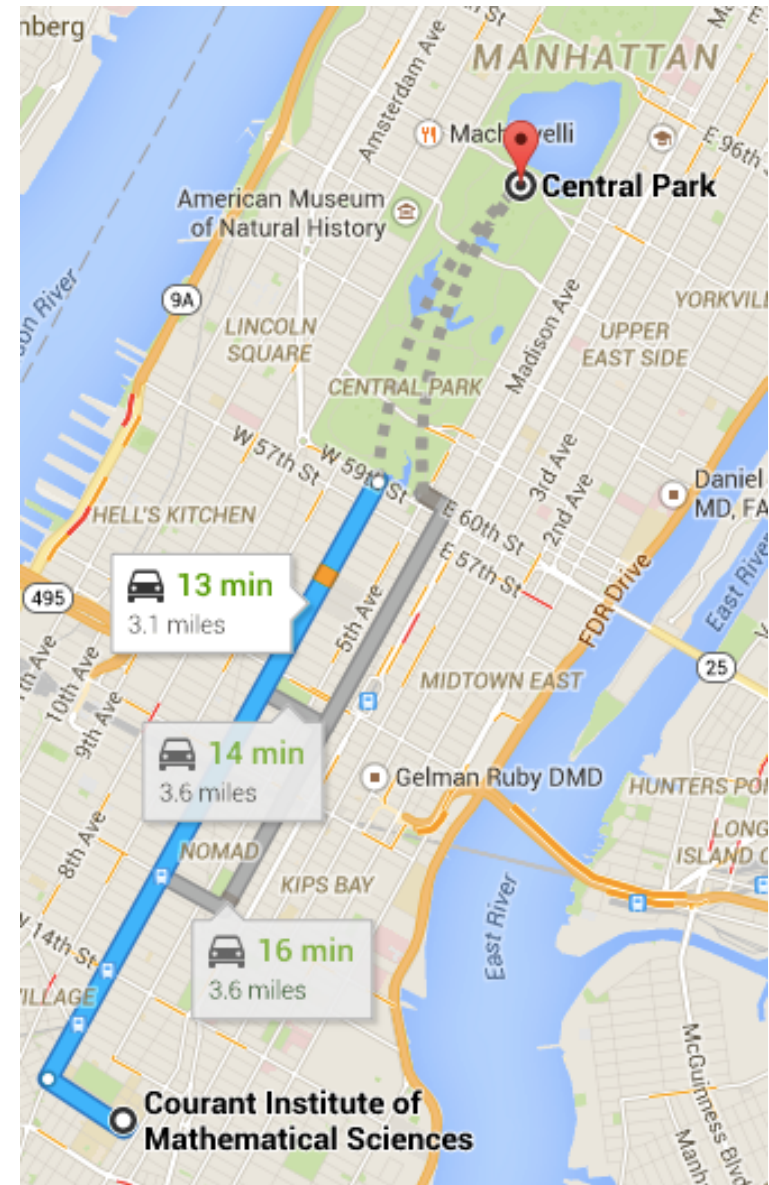
$$R_T = O(\sqrt{T \log N}).$$

→ informative even for N very large.

- Problem: computational complexity of algorithm in $O(N)$.
Can we derive more efficient algorithms when experts admit some structure and when loss is decomposable?

Example: Online Shortest Path

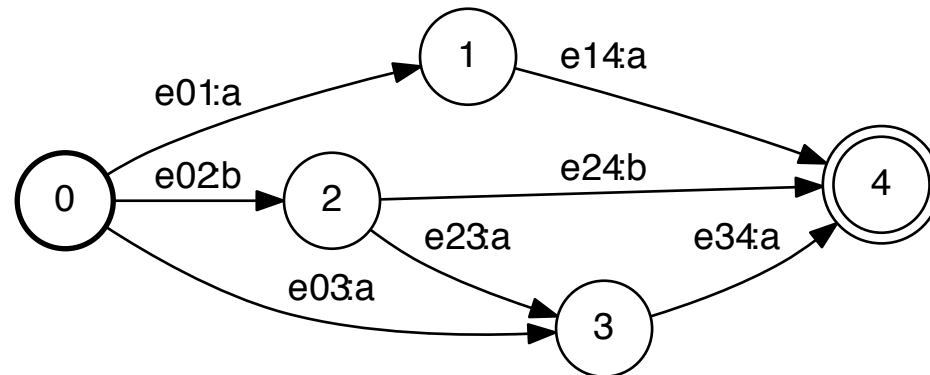
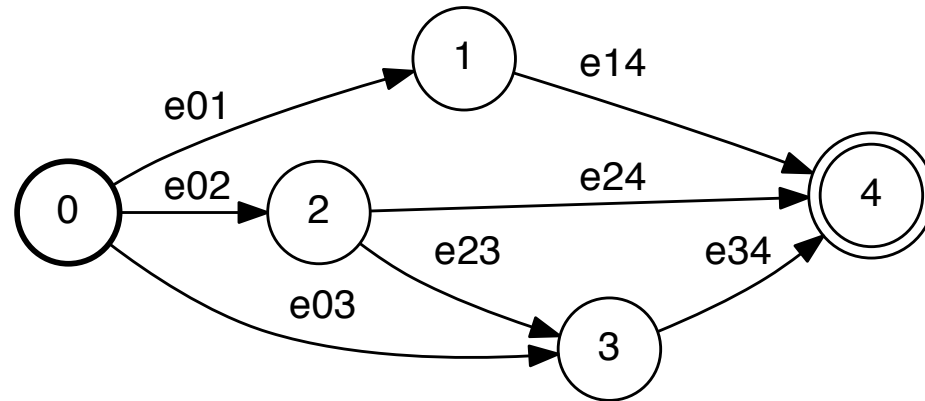
- **Problems:** path experts.
 - sending packets along paths of a network with routers (vertices); delays (losses).
 - car route selection in presence of traffic (loss).



Outline

- RWM with Path Experts
- FPL with Path Experts

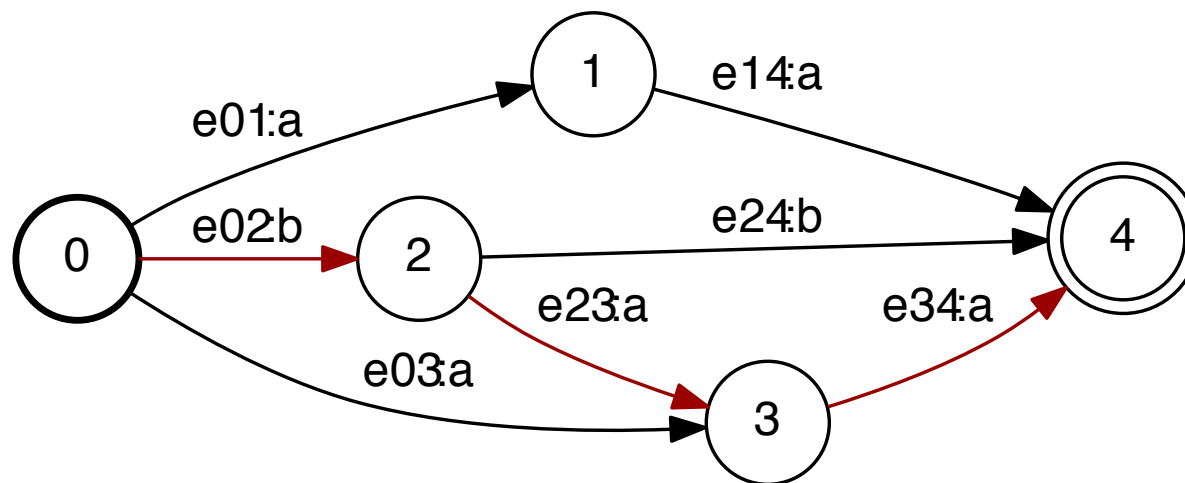
Path Experts



Additive Loss

- For path $\xi = e_{02}e_{23}e_{34}$,

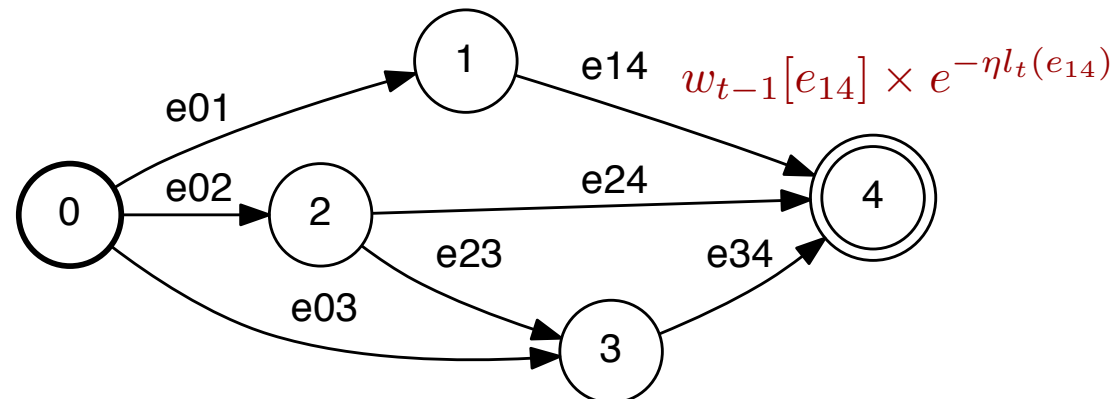
$$l_t(\xi) = l_t(e_{02}) + l_t(e_{23}) + l_t(e_{34}).$$



RWM + Path Experts

(Takimoto and Warmuth, 2002)

- **Weight update:** at each round t , update weight of path expert $\xi = e_1 \cdots e_n$:
 - $w_t[\xi] \leftarrow w_{t-1}[\xi] e^{-\eta l_t(\xi)}$; equivalent to
 - $w_t[e_i] \leftarrow w_{t-1}[e_i] e^{-\eta l_t(e_i)}$.



- **Sampling:** need to make graph/automaton stochastic.

Weight Pushing Algorithm

(MM 1997; MM, 2009)

- Weighted directed graph $G = (Q, E, w)$ with set of initial vertices $I \subseteq Q$ and final vertices $F \subseteq Q$:

- for any $q \in Q$,

$$d[q] = \sum_{\pi \in P(q, F)} w[\pi].$$

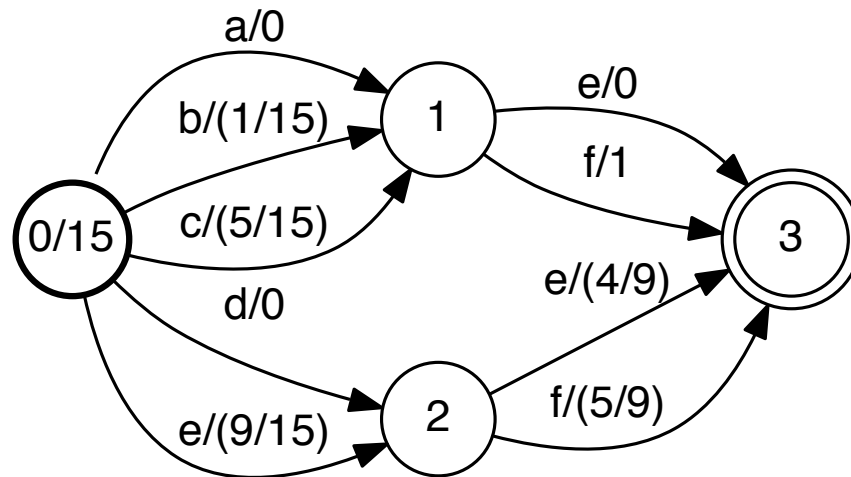
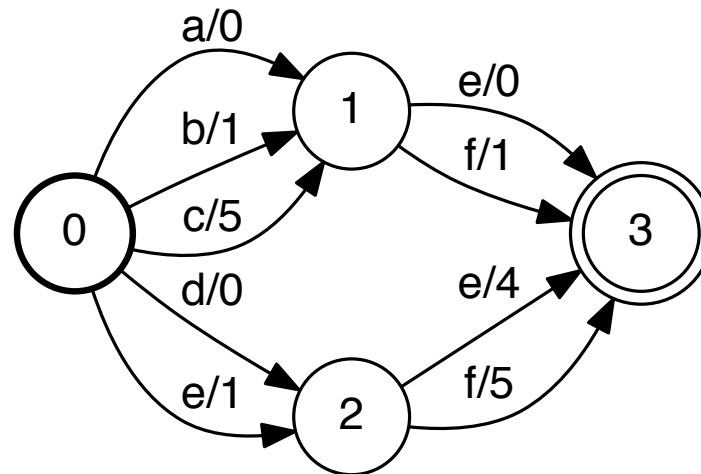
- for any $e \in E$ with $d[\text{orig}(e)] \neq 0$,

$$w[e] \leftarrow d[\text{orig}(e)]^{-1} \cdot w[e] \cdot d[\text{dest}(e)].$$

- for any $q \in I$, initial weight

$$\lambda(q) \leftarrow d(q).$$

Illustration



Properties

- **Stochasticity:** for any $q \in Q$ with $d[q] \neq 0$,

$$\sum_{e \in E[q]} w'[e] = \sum_{e \in E[q]} \frac{w[e] d[\text{dest}(e)]}{d[q]} = \frac{d[q]}{d[q]} = 1.$$

- **Invariance:** path weight preserved. Weight of path $\xi = e_1 \cdots e_n$ from I to F :

$$\begin{aligned} & \lambda(\text{orig}(e_1)) w'[e_1] \cdots w'[e_n] \\ &= d[\text{orig}(e_1)] \frac{w[e_1] d[\text{dest}(e_1)]}{d[\text{orig}(e_1)]} \frac{w[e_2] d[\text{dest}(e_2)]}{d[\text{dest}(e_1)]} \cdots \\ &= w[e_1] \cdots w[e_n] d[\text{dest}(e_n)] \\ &= w[e_1] \cdots w[e_n] = w[\xi]. \end{aligned}$$

Shortest-Distance Computation

■ Acyclic case:

- special instance of a generic single-source shortest-distance algorithm working with an arbitrary queue discipline and any k -closed semiring (MM, 2002).
- linear-time algorithm with the topological order queue discipline, $O(|Q| + |E|)$.

Generic Single-Source SD Algo.

(MM, 2002)

GEN-SINGLE-SOURCE(G, s)

```
1  for  $i \leftarrow 1$  to  $|Q|$  do
2       $d[i] \leftarrow r[i] \leftarrow \bar{0}$ 
3   $d[s] \leftarrow r[s] \leftarrow \bar{1}$ 
4   $Q \leftarrow \{s\}$ 
5  while  $Q \neq \emptyset$  do
6       $q \leftarrow \text{HEAD}(Q)$ 
7       $\text{DEQUEUE}(Q)$ 
8       $r' \leftarrow r[q]$ 
9       $r[q] \leftarrow \bar{0}$ 
10     for each  $e \in E[q]$  do
11         if  $d[n[e]] \neq d[n[e]] \oplus (r' \otimes w[e])$  then
12              $d[n[e]] \leftarrow d[n[e]] \oplus (r' \otimes w[e])$ 
13              $r[n[e]] \leftarrow r[n[e]] \oplus (r' \otimes w[e])$ 
14         if  $n[e] \notin Q$  then
15              $\text{ENQUEUE}(Q, n[e])$ 
```

Shortest-Distance Computation

■ General case:

- all-pairs shortest-distance algorithm in $(+, \times)$; for all pairs of vertices (p, q) ,

$$d[p, q] = \sum_{\pi \in P(p, q)} w[\pi].$$

- generalization of Floyd-Warshall algorithm to non-idempotent semirings (MM, 2002).
- time complexity in $O(|Q|^3)$, space complexity in $O(|Q|^2)$.
- alternative: approximation using generic single-source shortest-distance algorithm (MM, 2002).

Generic All-Pairs SD Algorithm

(MM, 2002)

GEN-ALL-PAIRS(G)

```
1  for  $i \leftarrow 1$  to  $|Q|$  do
2      for  $j \leftarrow 1$  to  $|Q|$  do
3           $d[i, j] \leftarrow \bigoplus_{e \in E \cap P(i, j)} w[e]$ 
4  for  $k \leftarrow 1$  to  $|Q|$  do
5      for  $i \leftarrow 1$  to  $|Q|, i \neq k$  do
6          for  $j \leftarrow 1$  to  $|Q|, j \neq k$  do
7               $d[i, j] \leftarrow d[i, j] \oplus (d[i, k] \otimes d[k, k]^* \otimes d[k, j])$ 
8      for  $i \leftarrow 1$  to  $|Q|, i \neq k$  do
9           $d[k, i] \leftarrow d[k, k]^* \otimes d[k, i]$ 
10          $d[i, k] \leftarrow d[i, k] \otimes d[k, k]^*$ 
11          $d[k, k] \leftarrow d[k, k]^*$ 
```

In-place version.

Learning Guarantee

- **Theorem:** let \mathcal{N} be total number of path experts and M an upper bound on the loss of a path expert. Then, the (expected) regret of RWM is bounded as follows:

$$\mathcal{L}_T \leq \mathcal{L}_T^{\min} + 2M\sqrt{T \log \mathcal{N}}.$$

Exponentiated Weighted Avg

- Computation of the prediction at each round:

$$\hat{y}_t = \frac{\sum_{\xi \in P(I, F)} w_t[\xi] y_{t, \xi}}{\sum_{\xi \in P(I, F)} w_t[\xi]}.$$

- Two single-source shortest-distance computations:
 - edge weight $w_t[e]$ (denominator).
 - edge weight $w_t[e] y_t[e]$ (numerator).

FPL + Path Experts

- Weight update: at each round, update weight of edge e ,

$$w_t[e] \leftarrow w_{t-1}[e] + l_t(e).$$

- Prediction: at each round, shortest path after perturbing each edge weight:

$$w'_t[e] \leftarrow w_t[e] + p_t(e),$$

where $\mathbf{p}_t \sim U([0, 1/\epsilon]^{|E|})$

or $\mathbf{p}_t \sim \text{Laplacian}$ with density $f(\mathbf{x}) = \frac{\epsilon}{2} e^{-\epsilon \|\mathbf{x}\|_1}$.

Learning Guarantees

■ **Theorem:** assume that edge losses are in $[0, 1]$. Let l_{\max} be the length of the longest path from I to F and M an upper bound on the loss of a path expert. Then,

- the (expected) regret of FPL is bounded as follows:

$$\mathbb{E}[R_T] \leq 2\sqrt{l_{\max} M |E| T} \leq 2l_{\max} \sqrt{|E| T}.$$

- the (expected) regret of FPL* is bounded as follows:

$$\begin{aligned} \mathbb{E}[R_T] &\leq 4\sqrt{\mathcal{L}_T^{\min} |E| l_{\max} (1 + \log |E|)} + 4|E| l_{\max} (1 + \log |E|) \\ &\leq 4l_{\max} \sqrt{T |E| (1 + \log |E|)} + 4|E| l_{\max} (1 + \log |E|) \\ &= O(l_{\max} \sqrt{T |E| \log |E|}). \end{aligned}$$

Proof

- For FPL, use bound of previous lectures with

$$X_1 = |E| \quad W_1 = l_{\max} \quad R = M \leq l_{\max}.$$

- For FPL*, use bound of previous lecture with

$$X_1 = |E| \quad W_1 = l_{\max} \quad N = |E|.$$

Computational Complexity

- For an acyclic graph:
 - T updates of all edge weights.
 - T runs of a linear-time single-source shortest-path.
 - overall $O(T(|Q| + |E|))$.

Extensions

- Component hedge algorithm (Koolen, Warmuth, and Kivinen, 2010):
 - optimal regret complexity: $R_T = O(M \sqrt{T \log |E|})$.
 - special instance of mirror descent.
- Non-additive losses (Cortes, Kuznetsov, MM, Warmuth, 2015):
 - extensions of RWM and FPL.
 - rational and tropical losses.

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