SOME PROBLEMS ON THE PRIME FACTORS OF CONSECUTIVE INTEGERS II

by

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G. A. Grimm [3] stated the following interesting conjecture: Let $n + 1, \ldots, n + k$ be consecutive composite numbers. Then for each $i$, $1 \leq i \leq k$ there is a $p_i$, $p_i \mid n + i$ $p_{i_1} \neq p_{i_2}$ for $i_1 \neq i_2$. He also expressed the conjecture in a weaker form stating that any set of $k$ consecutive composite numbers need to have at least $k$ prime factors. We first show that even in this weaker form the conjecture goes far beyond what is known about primes at present.

First we define a few number theoretic functions. Denote by $v(n, k)$ the number of distinct prime factors of $(n + 1)\ldots(n + k)$. $f_1(n)$ is the smallest integer $k$ so that for every $1 \leq i \leq k$

$$v(n, k) \geq i \text{ but } v(n, k + 1) = k.$$ 

$f_0(n)$ is the largest integer $k$ for which

$$v(n, k) \geq k.$$ 

Clearly $f_0(n) \geq f_1(n)$ and we shall show that infinitely often $f_0(n) > f_1(n)$.

Following Grimm let $f_2(n)$ be the largest integer $k$ so that for each $1 \leq i \leq k$ there is a $p_i \mid n + i$, $p_{i_1} \neq p_{i_2}$ if $i_1 \neq i_2$.

Denote by $P(m)$ the greatest prime factor of $m$. $f_3(n)$ is the greatest integer so that all the primes $P(n + i)$, $1 \leq i \leq k$ are distinct. $f_4(n)$ is the largest integer $k$ so that $P(n + i) \geq i$, $1 \leq i \leq k$ and $f_5(n)$ is the largest integer $k$ so that $P(n + i) \geq k$ for every $1 \leq i \leq k$. Clearly
\[ f_0(n) \geq f_1(n) \geq f_2(n) \geq f_3(n) \geq f_4(n) \geq f_5(n). \]

**CONJECTURE:** It seems certain to us that for infinitely many \( n \) the inequalities are all strict. For example, for \( n = 9701 \)

\[ f_0(n) = 96 > 94 > 90 > 45 > 18 > 11 = f_5(n). \]

It seems very difficult to get exact information on these functions which probably behave very irregularly. By a well known theorem of Pólya, \( f_3(n) \) tends to infinity. First we prove

**THEOREM 1.**

\[ f_0(n) < c_1 \left( \frac{n}{\log n} \right)^{1/2} \]

To prove (1) assume that \( \nu(n, k) \geq k \). We then would have

\[ \left( \frac{n+k}{k} \right)^2 \Pi p_r, \pi(k) < r \leq k \]

where \( p_1 = 2 < p_2 < \ldots \) is the sequence of consecutive primes. On the other hand

\[ \left( \frac{n+k}{k} \right)^k < \left( \frac{n+k}{k} \right)^k < \left( \frac{e(n+k)}{k} \right)^k. \]

A well known theorem of Rosser and Schoenfeld states that for large \( t \)

\[ p_\pi(t) > t \log t + t \log \log t - c_2 t \]

where \( c_1, c_2 \ldots \) are positive absolute constants.

From (4) we obtain by a simple computation that \( (e^z = e^z) \).

\[ \prod_{r=\pi(k)+1}^k p_r > \exp (k \log k + k \log \log k - c_3 k). \]
From (2), (3), (4) and (5) we have

$$e^{(n+1)/k} > k \log k / e^{c_3}.$$  

(6) immediately implies (1) and the proof of Theorem 1 is complete.

We conjecture

$$f_0(n) < n^{1/2-c_4}$$

for all $n > n_0(c_4)$, perhaps $f_1(n) > n^{c_5}$ for all $n$. $f_0(n) < n^{1/2-c_4}$ seems to follow from a recent result of Ramachandra (A note on numbers with a large prime factor, Journal London Math. Soc. 1 (1969), pp. 303-306) but we do not give the details here.

Theorem 1 shows that there is not much hope to prove Grimm's conjecture in the "near future" since even its weaker form implies that

$$p_{i+1} - p_i < c(p_i / \log p_i)^{1/2}$$
in particular it would imply that there are primes between $n^2$ and $(n+1)^2$ for all sufficiently large $n$.

Next we show

THEOREM 2. For infinitely many $n$

$$f_0(n) < c_6 n^{1/e} \quad \text{and} \quad f_1(n) < c_7 n^{1/e}.$$  

Denote by $u(m, X)$ the number of prime factors of $m$ in $(c_{8X}^{1/e}, X)$. We evidently have

$$\sum_{m=1}^{X} u(m, X) = \sum_{1/e}^{X} \left[ \frac{X}{p} \right] > X \sum_{c_{8X}^{1/e} < p < X} \frac{1}{p} - \pi(X) > X \sum_{c_{8X}^{1/e} < p < X} \frac{1}{p}$$

for sufficiently small $c_8$. 
From (7) it is easy to see that there is an $c_0X^{1/e} < m < X = c_0X^{1/e}$
so that for every $t \leq X - m$

$$\sum_{i=1}^{t} u(m + i, X) \geq t.$$ 

Choose $t = c_0X^{1/e}$ and we obtain Theorem 2. In fact for every $t < c_0X^{1/e}$
$\prod_{i=1}^{t} (m + i)$ has at least $t$ prime factors $> c_0X^{1/e}$. The same method gives
that $f_1(n) < c_0n^{1/e}$ holds for infinitely many $n$.

We can improve a result of Grimm by

**Theorem 3.** For every $n > n_0$

$$f_2(n) > (1 + o(1)) \log n.$$ 

Suppose $f_2(n) < t$. This implies by Hall's theorem that for some $r \leq \pi(t)$
there are $r$ primes $p_1, \ldots, p_r$ so that $r + 1$ integers $n + i_1, \ldots, n + i_{r+1}$,
$1 \leq i_1 < \ldots < i_{r+1} < t$ are entirely composed of $p_1, \ldots, p_r$. For each $p$
there is at most one of the integers $n + j, 1 < j < t$ which divide $p^\alpha$
with $p^\alpha > t$. Thus for at least one index $i_s, 1 \leq s \leq r + 1$

$$n + i_s = \prod_{i=1}^{t} p_i^{\alpha_i}, \quad p_i < t, \text{ or } n < t^{\pi(r)} < t^{\pi(t)} < e^{(1+o(1))t}$$ 

which proves Theorem 3. Probably this proof can be improved to give

$$f_2(n) / \log n \to \infty$$ 

but at the moment we can not see how to get

$$f_2(n) > (\log n)^{1+\epsilon}.$$ 

Probably

$$f_2(n) / (\log n)^k \to \infty$$

for every $k$ which would make Grimm's conjecture likely in view of the fact
that "probably"

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* K. Ramachandra just informed us that he can prove $f_2(n) > c \log n (\log \log n)^{1/4}$
\[
\lim (p_{r+1} - p_r) / (\log p_r)^k \to 0
\]

for sufficiently large \( k \). We certainly do not see how to prove (8) but this may be due to the fact that we overlook a simple idea. On the other hand the proof of (9) seems beyond human ingenuity at present.

In view of [2]
\[
\lim \frac{p_{r+1} - p_r}{\log p_r} < 1.
\]

Theorem 3 shows that Grimm's conjecture holds for infinitely many sets of composite numbers between consecutive primes.

**THEOREM 4.** For infinitely many \( n \)
\[
f_3(n) > \exp(c_9 (\log n \log \log n)^{1/2}).
\]

A well known theorem of de Bruijn [1] implies that for an absolute constant \( c_9 \) the number of integers \( m < n \) for which
\[
P(m) < \exp(c_9 (\log n \log \log n)^{1/2})
\]
is less than
\[
n \exp(-c_9 (\log n \log \log n)^{1/2}).
\]

(10) and (11) imply that there are \( \exp(c_9 (\log n \log \log n)^{1/2}) \) consecutive integers not exceeding \( n \) all of whose greatest prime factors are greater than \( \exp(c_9 (\log n \log \log n)^{1/2}) \), which proves Theorem 4.

It seems likely that for infinitely many \( n \)
\[
f_3(n) < (\log n)^{c_{10}}
\]
but it is quite possible that for all \( n \)
\[
f_3(n) > (\log n)^{c_{11}}.
\]

We have no non-trivial upper bounds for \( f_3(n), f_4(n) \) or \( f_5(n) \). It seems certain that \( f_3(n) = o(n^\epsilon) \) for every \( \epsilon > 0 \). It is difficult to guess good upper or lower bounds for \( f_2(n) \).
Grimm observed that there are integers $u$ and $v$, $u < v, P(u) = P(v)$ so that there is no prime between $u$ and $v$ e.g. $u = 24, v = 27$. It is easy to find many other such examples, but we cannot prove that there are infinitely many such pairs $u_1, v_1$ and we cannot get good upper or lower bounds for $v_1 - u_1$. Pólya's theorem of course implies $v_1 - u_1 \to \infty$.

It has been conjectured (at the present we cannot trace the conjecture) that if $n_1$ and $m_1$ have the same prime factors, then there is always a prime between $n_1$ and $m_1$. We cannot get good upper or lower bounds on $m_1 - n_1$.

Next we prove

**THEOREM 5.** Each of the inequalities

$$f_i(n) > f_{i+1}(n), 0 \leq i \leq 4$$

have infinitely many solutions.

First we prove $f_0(n) > f_1(n)$ infinitely often. Put $n = pq$

where $p$ and $q$ are distinct primes, $q = (1 + o(1)) p$, i.e. $p$ and $q$ are both of the form $(1 + o(1)) n^{1/2}$. There is a largest $k$ for which

(12)

$$f_0(pq - k) \geq k.$$

By theorem 1 none of the integers $pq - 1, \ldots, pq - k + 1$ can be multiples of $p$ or $q$ since $k = o(n^{1/2})$. Since $k$ is maximal, by (12) the number of distinct prime factors of the product $(pq - k + 1)$ equals $k$. Thus the number of distinct prime factors of $(pq - k + 1)\ldots(pq - 1)$ is $k - 2$ hence $f_1(pq - k) < k - 1$ while $f_0(pq - k) \geq k$. 

To prove $f_1(n) > f_2(n)$ infinitely often, observe that
\[ f_1(pq - 1) > f_2(pq - 1) \] with $p$ and $q$ as above. Since
\[ f_1(pq - 1) > \min(p, q) , \] the primes $p$ and $q$ cannot both be used for
$f_2$ but can be used for $f_1$. Assume now $f_2(n) = k$ and assume that the set
$n + 1, \ldots, n + k$
contains no power of a prime. Then $f_2(n) > f_3(n)$. Since $f_2(n) = k$
there must be $r$ numbers $n + i_1, \ldots, n + i_r$ in the set which to-
gether with $n + k + 1$ are composed entirely of exactly $r$ primes
$q_1 < \ldots < q_r$ (we use Hall's theorem). Now none of these $r$ numbers
is a power of $q_1$ so their largest prime factors cannot all be distinct
and thus $f_3(n) < k$.

Now clearly $n^2$ and $(n + 1)^2$ infinitely often have no power
between them. This and the fact that $f_2(n^2) = o(n)$ gives infinitely
often $f_2(n^2) > f_3(n^2)$. It might be interesting to try to determine the largest
$n$ such that $f_2(n) = f_3(n)$. We cannot even prove there is such an $n$.

Since $f_3(n)$ goes to infinity with $n$ and $f_4(2^k - 3) =
f_3(2^k - 3) = 2$, it is clear that $f_3(n) > f_4(n)$ infinitely often.
Also $f_4(2^k - 1) > 2$ if $k > 1$ while $f_3(2^k - 1) = 2$. In fact it is
easy to see that $f_4(2^k - 1)$ goes to infinity with $k$.

**THEOREM 6.** For all $n > n_0$, $f_1(n) > f_3(n)$.

**Proof:** Put $f_1(n) = k$. Then $(n + 1) \ldots (n + k)$ has exactly $k$
distinct prime factors. If $f_3(n) = k$ then all these $k$ primes must
be the greatest prime factor of some $n + i, 1 \leq i \leq k$. In particular
$2$ must be the greatest prime factor of $n + i$, $(n + i = 2^m)$ and
similarly for $3$ so that $n + i_2 = 2^v 3^w$. 
Thus by theorem 1

(13) \[ |2^u - 2^v z^w| < k < 2^{u/2} \]

A well known theorem states that if \( p_1, \ldots, p_r \) are \( r \) given primes and \( a_1 < a_2 < \ldots \) is the set of integers composed of the \( p \)'s
then \( a_{i+1} - a_i > a_1^{1-\epsilon} \) for every \( \epsilon > 0 \) and \( i > i(\epsilon) \). This clearly contradicts (13), proving theorem 6.

It is not impossible that for every \( n > n_0 \)

\[ f_0(n) > f_1(n) > f_2(n) > f_3(n) > f_4(n) \]

but we are far from being able to prove this. It seems certain to us that

\( f_1(n) > f_2(n) > f_3(n) \) for all \( n > n_0 \) but we might hazard the guess

that \( f_0(n) = f_1(n) \) infinitely often, and perhaps \( f_3(n) = f_4(n) = f_5(n) \)

infinitely often. \( f_4(2^k - 3) = f_5(2^k - 3) = 2 \), thus \( f_4(n) = f_5(n) \) has

infinitely many solutions.

We can prove by using the methods of Theorem 4 that

\[ f_3(n) < \exp((2 + o(1))(\log n \log \log n)^{1/2}) \]

for infinitely many \( n \) and that

\[ f_2(n) < \exp(e \log n \log \log n / \log \log n) \]

for infinitely many \( n \).

Perhaps our methods give that \( f_0(n) < n^{1/e} \) holds infinitely

often and perhaps \( f_0(n) < n^{\frac{1}{e+\epsilon}} \) holds for every \( n > n_0 \). All these

and related questions we hope to investigate.
References

1. N.G. de Bruijn, On the number of positive integers ≤ X and free of prime factors > y. Indig. Math. 13 (1951) 50-60.


